

Infrared divergences at NNLO

Stefan Weinzierl

Università di Parma

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The need for NNLO calculations

Hunting for the Higgs and other yet-to-be-discovered particles requires a better knowledge of the theoretical cross section.

The strong coupling constant α_s is one fundamental parameter of the theory and its precise value affects the magnitude of many (background) processes.

The next generation of colliders will increase the experimental precision. This has to be matched by an improvement in the accuracy of theoretical predictions.

Theoretical predictions are calculated as a power expansion in the coupling. Higher precision is reached by including the next higher term in the perturbative expansion.

What is necessary:

NNLO calculations

Perturbative NNLO calculations

The experimental needs are numerical programs which yield predictions for a wide range of observables.

Fully differential NNLO programs for

- Bhabha scattering
- $pp \rightarrow 2$ jets
- $e^+e^- \rightarrow 3$ jets

which allow the calculation of any infrared safe observable.

Infrared safe at NNLO:

$$\text{Single unresolved} : \mathcal{O}_{n+1}(p_1, \dots, p_{n+1}) \rightarrow \mathcal{O}_n(p'_1, \dots, p'_n),$$

$$\text{Double unresolved} : \mathcal{O}_{n+2}(p_1, \dots, p_{n+2}) \rightarrow \mathcal{O}_n(p'_1, \dots, p'_n).$$

Necessary ingredients for a NNLO calculation

- Calculation of the (two-loop) amplitudes.

Requires: Two-loop integrals and tensor reduction.

- Cancellation of IR divergences has to be done before any Monte Carlo integration.

Requires: Extension of the subtraction or slicing method to NNLO.

- The final numerical computer program.

Requires: Stable and efficient numerical methods.

Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, **loop integrals** can **have infrared divergences**.

For each IR divergence there is a **corresponding divergence with the opposite sign** in the real emission amplitude, when particles becomes **soft** or **collinear** (e.g. unresolved).



The **Kinoshita-Lee-Nauenberg** theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

General methods at NLO

Fully differential **NLO Monte Carlo programs** need a general method to handle the cancelation of infrared divergencies.

- **Phase space slicing**

- e^+e^- : W. Giele and N. Glover, (1992)
- **initial hadrons**: W. Giele, N. Glover and D.A. Kosower, (1993)
- **massive partons, fragmentation**: S. Keller and E. Laenen, (1999)

- **Subtraction method**

- **residue approach**: S. Frixione, Z. Kunzst and A. Signer, (1995)
- **dipole formalism**: S. Catani and M. Seymour, (1996)
- **massive partons**: L. Phaf and S.W. (2001),
S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

The dipole formalism at NLO

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\begin{aligned}\sigma^{NLO} &= \int_{n+1} d\sigma^R + \int_n d\sigma^V \\ &= \int_{n+1} (d\sigma^R - d\sigma^A) + \int_n \left(d\sigma^V + \int_1 d\sigma^A \right)\end{aligned}$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the **same pointwise singular behaviour in D dimensions** as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \rightarrow 0$.
- **Analytic integrability in D dimensions** over the one-parton subspace leading to soft and collinear divergences.

The subtraction method at NNLO

Contributions at NNLO:

$$d\sigma_{n+2}^{(0)} = \left(\mathcal{A}_{n+2}^{(0)*} \mathcal{A}_{n+2}^{(0)} \right) d\phi_{n+2},$$

$$d\sigma_{n+1}^{(1)} = \left(\mathcal{A}_{n+1}^{(0)*} \mathcal{A}_{n+1}^{(1)} + \mathcal{A}_{n+1}^{(1)*} \mathcal{A}_{n+1}^{(0)} \right) d\phi_{n+1},$$

$$d\sigma_n^{(2)} = \left(\mathcal{A}_n^{(0)*} \mathcal{A}_n^{(2)} + \mathcal{A}_n^{(2)*} \mathcal{A}_n^{(0)} + \mathcal{A}_n^{(1)*} \mathcal{A}_n^{(1)} \right) d\phi_n,$$

Adding and subtracting:

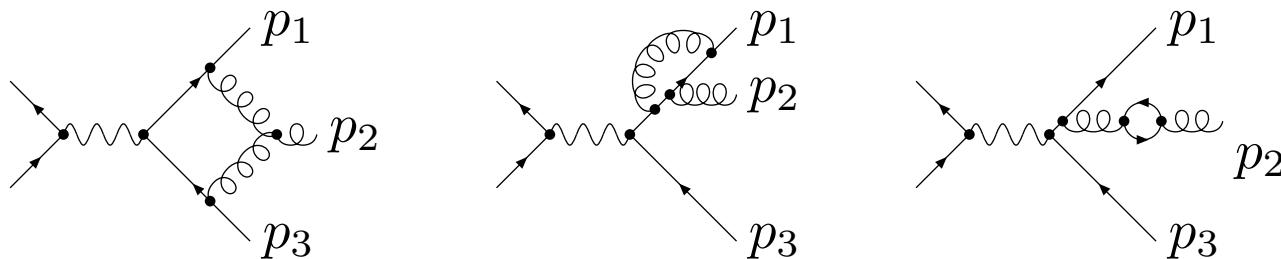
$$\begin{aligned} \langle \mathcal{O} \rangle_n^{NNLO} &= \int \left(\mathcal{O}_{n+2} d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_n^{(0,2)} \right) \\ &+ \int \left(\mathcal{O}_{n+1} d\sigma_{n+1}^{(1)} + \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_n^{(1,1)} \right) \\ &+ \int \left(\mathcal{O}_n d\sigma_n^{(2)} + \mathcal{O}_n \circ d\alpha_n^{(0,2)} + \mathcal{O}_n \circ d\alpha_n^{(1,1)} \right). \end{aligned}$$

Colour disentanglement

Full QCD amplitudes do not factorize in the singular limits, but **primitive amplitudes** do.

Primitive amplitudes distinguished by:

- **fixed cyclic ordering** of the QCD partons
- definite **routing of the external fermion lines** through the diagram
- **particle content circulating in the loop**



The leading singular behaviour

Factorization of **one-loop amplitudes** in **single unresolved limits**:

$$A_n^{(1)} = \text{Sing}^{(0,1)} \cdot A_{n-1}^{(1)} + \text{Sing}^{(1,1)} \cdot A_{n-1}^{(0)},$$

Z. Bern, Del Duca, W. Kilgore, C. Schmidt, D. Kosower, P. Uwer, S. Catani, M. Grazzini

Factorization of **tree amplitudes** in **double unresolved limits**:

$$A_n^{(0)} = \text{Sing}^{(0,2)} \cdot A_{n-2}^{(0)},$$

F. Berends, W. Giele, J. Cambell, N. Glover, S. Catani, M. Grazzini, V. Del Duca, A. Frizzo, F. Maltoni, D. Kosower

For the subtraction method one needs also the subleading singular behaviour !

An example involving double unresolved configurations

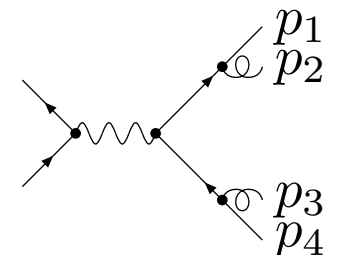
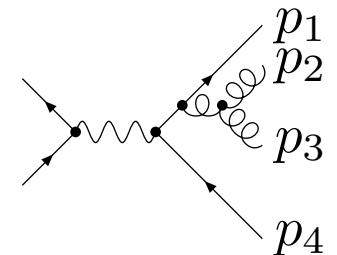
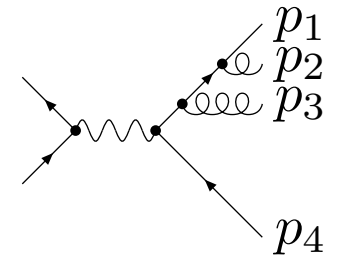
The leading-colour contributions to $e^+e^- \rightarrow qgg\bar{q}$.

Double unresolved configurations:

- Two pairs of separately collinear particles
- Three particles collinear
- Two particles collinear and a third soft particle
- Two soft particles
- Coplanar degeneracy

Single unresolved configurations:

- Two collinear particles
- One soft particle



Results

$$\begin{aligned}
 \mathcal{P}_{(0,0)}^{(0,2)} q \rightarrow qgg \left(\begin{array}{c} \text{Diagram} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right) &= p_e^{(2)} \frac{4}{s_{1234}^2} \frac{1}{x_1^2} \left\{ \frac{1}{y_1 (y_1 + \bar{y}_1 \bar{z}_1)} \left[\frac{2}{(y_1 + \bar{y}_1 \bar{z}_1) x_1 + \bar{y}_1 x_2} + \frac{2}{y_1 x_1 + u} \right. \right. \\
 &+ \left. \frac{u}{(y_1 + \bar{y}_1 \bar{z}_1) x_1 + \bar{y}_1 x_2} (-2 + (1 - \varepsilon)u) + \frac{\bar{y}_1 x_2}{u} (-2 + (1 - \varepsilon)\bar{y}_1 x_2) \right] \\
 &+ \frac{1}{y_1} \left[\frac{2}{u [(y_1 + \bar{y}_1 \bar{z}_1) x_1 + \bar{y}_1 x_2]} + \frac{1}{x_1 + \bar{y}_1 x_2} (-2 + (1 - \varepsilon)u) - \frac{4}{u} + (1 - \varepsilon) \frac{\bar{y}_1 x_2}{u} - \frac{\bar{y}_1 x_2}{u} (-2 + (1 - \varepsilon)\bar{y}_1 x_2) \right. \\
 &+ \varepsilon(1 - \varepsilon)u - \varepsilon(1 - \varepsilon)\bar{y}_1 x_2 - 1 + 3\varepsilon] + (1 - \varepsilon)^2 \frac{\bar{y}_1 \bar{z}_1}{y_1} + 3 - 4\varepsilon + \varepsilon^2 + \frac{x_1}{y_1} \left[-\varepsilon \left(\frac{2}{y_1 + \bar{y}_1 \bar{z}_1} - 2 + (1 - \varepsilon)\bar{y}_1 \bar{z}_1 \right) \right. \\
 &\left. \left. + \frac{2}{u} - (1 - \varepsilon) \left(\frac{x_1}{u} - 2\bar{y}_1 z_1 + (1 - \varepsilon)\bar{y}_1 z_1 x_1 \right) + \frac{1}{u (y_1 + \bar{y}_1 \bar{z}_1)} (-2 + (1 - \varepsilon)x_1) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{P}_{(0,1)}^{(0,2)} q \rightarrow qgg \left(\begin{array}{c} \text{Diagram} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right) &= p_e^{(2)} \frac{4}{s_{1234}^2} \frac{1}{x_1^2} \left[\frac{2}{x_1 + x_2} - 2 + (1 - \varepsilon) x_2 - \varepsilon x_1 \right] \\
 &\times \frac{1}{y_1} \left[\frac{2}{1 - z_1(1 - y_1)} - 2 + (1 - \varepsilon) (1 - y_1)(1 - z_1) \right]
 \end{aligned}$$

The subtraction terms for one-loop amplitudes with one unresolved parton

Example: The splitting $q \rightarrow qg$.

$$\mathcal{P}_{(1,0) q \rightarrow qg, lc, corr}^{(1,1)} = S_\epsilon^{-1} c_\Gamma \left(\frac{-s_{ijk}}{\mu^2} \right)^{-\epsilon} y^{-\epsilon} \\ \times \left\{ g_{1,corr}(y, z) \mathcal{P}_{q \rightarrow qg}^{(0,1)} + f_2 \frac{2}{s_{ijk}} \frac{1}{y} \not{p}_e [1 - \rho\epsilon(1-y)(1-z)] \right\} - \frac{11}{6\epsilon} \mathcal{P}_{q \rightarrow qg}^{(0,1)},$$

$$\mathcal{P}_{(1,0) q \rightarrow qg, nf}^{(1,1)} = \frac{1}{3\epsilon} \mathcal{P}_{q \rightarrow qg}^{(0,1)},$$

$$\mathcal{P}_{(1,0) q \rightarrow qg, sc}^{(1,1)} = S_\epsilon^{-1} c_\Gamma \left(\frac{-s_{ijk}}{\mu^2} \right)^{-\epsilon} y^{-\epsilon} \\ \times \left\{ \left[-\frac{1}{\epsilon^2} - g_{1,intr}(y, 1-z) \right] \mathcal{P}_{q \rightarrow qg}^{(0,1)} + f_2 \frac{2}{s_{ijk}} \frac{1}{y} \not{p}_e [1 - \rho\epsilon(1-y)(1-z)] \right\}.$$

with

$$g_{1,intr}(y, z) = -\frac{1}{\epsilon^2} \left[\Gamma(1+\epsilon)\Gamma(1-\epsilon) \left(\frac{z}{1-z} \right)^\epsilon + 1 - (1-y)^\epsilon z^\epsilon {}_2F_1(\epsilon, \epsilon, 1+\epsilon; (1-y)(1-z)) \right].$$

Integration over the unresolved phase space

Reduction to basic integrals:

$$\int_0^1 dy y^a (1-y)^{1+c+d} \int_0^1 dz z^c (1-z)^d [1-z(1-y)]^e {}_2F_1(\varepsilon, \varepsilon; 1+\varepsilon; (1-y)z) =$$
$$\frac{\Gamma(1+a)\Gamma(1+d)\Gamma(2+a+d+e)\Gamma(1+\varepsilon)}{\Gamma(2+a+d)\Gamma(\varepsilon)\Gamma(\varepsilon)} \sum_{j=0}^{\infty} \frac{\Gamma(j+\varepsilon)\Gamma(j+\varepsilon)\Gamma(j+1+c)}{\Gamma(j+1)\Gamma(j+1+\varepsilon)\Gamma(j+3+a+c+d+e)},$$

Result proportional to **hypergeometric functions ${}_4F_3$ with unit argument.**

These **sums** are then **expanded into a Laurent series in ε** with the help of Z -sums

(S. Moch, P. Uwer, S.W.).

Simplifications: Relations among multiple-zeta values.

$$\zeta_2 = \frac{\pi^2}{6}, \quad \zeta_4 = \frac{\pi^4}{90}, \quad \zeta_{1,2} = \zeta_3, \quad \zeta_{1,3} = \frac{\pi^4}{360}, \quad \zeta_{2,2} = \frac{\pi^4}{120}, \quad \zeta_{1,1,2} = \frac{\pi^4}{90}.$$

Results for one-loop amplitudes with one unresolved parton

$\mathcal{V}^{(1,1)}$ defined by

$$8\pi^2 S_\varepsilon^{-1} \mu^{2\varepsilon} \int d\phi_{\text{unresolved}} \mathcal{P}^{(1,1)} = \frac{S_\varepsilon^{-2} (4\pi)^{2\varepsilon}}{\Gamma(1-\varepsilon)^2} \left(\frac{s_{ijk}}{\mu^2} \right)^{-2\varepsilon} \mathcal{T} \mathcal{V}^{(1,1)}.$$

For the splitting $q \rightarrow qg$ one finds with $L = \ln(s_{ijk}/\mu^2)$:

$$\begin{aligned} \mathcal{V}_{(1,0) q \rightarrow qg, lc, intr}^{(1,1)} &= -\frac{1}{4\varepsilon^4} - \frac{31}{12\varepsilon^3} + \left(-\frac{51}{8} - \frac{1}{4}\rho + \frac{5}{12}\pi^2 - \frac{11}{6}L \right) \frac{1}{\varepsilon^2} + \left(-\frac{151}{6} - \frac{55}{24}\rho + \frac{145}{72}\pi^2 + \frac{15}{2}\zeta_3 - \frac{11}{4}L - \frac{11}{12}L^2 \right) \frac{1}{\varepsilon} \\ &\quad - \frac{1663}{16} - \frac{233}{24}\rho + \frac{107}{16}\pi^2 + \frac{5}{12}\rho\pi^2 + \frac{356}{9}\zeta_3 - \frac{1}{72}\pi^4 - \frac{187}{24}L - \frac{11}{12}\rho L + \frac{55}{72}\pi^2 L - \frac{11}{8}L^2 - \frac{11}{36}L^3 \\ &\quad + i\pi \left[-\frac{1}{4\varepsilon^3} - \frac{3}{4\varepsilon^2} + \left(-\frac{29}{8} - \frac{1}{4}\rho + \frac{\pi^2}{3} \right) \frac{1}{\varepsilon} - \frac{139}{8} - \frac{11}{8}\rho + \pi^2 + \frac{15}{2}\zeta_3 \right] + \mathcal{O}(\varepsilon), \\ \mathcal{V}_{(1,0) q \rightarrow qg, nf}^{(1,1)} &= \frac{1}{3\varepsilon^3} + \left(\frac{1}{2} + \frac{1}{3}L \right) \frac{1}{\varepsilon^2} + \left(\frac{17}{12} + \frac{1}{6}\rho - \frac{5}{36}\pi^2 + \frac{1}{2}L + \frac{1}{6}L^2 \right) \frac{1}{\varepsilon} + \frac{33}{8} + \frac{7}{12}\rho - \frac{5}{24}\pi^2 - \frac{23}{9}\zeta_3 \\ &\quad + \frac{17}{12}L + \frac{1}{6}\rho L - \frac{5}{36}\pi^2 L + \frac{1}{4}L^2 + \frac{1}{18}L^3 + \mathcal{O}(\varepsilon), \\ \mathcal{V}_{(1,0) q \rightarrow qg, sc}^{(1,1)} &= \left(\frac{5}{8} - \frac{\pi^2}{6} \right) \frac{1}{\varepsilon^2} + \left(\frac{35}{8} + \frac{3}{8}\rho - \frac{\pi^2}{4} - 7\zeta_3 \right) \frac{1}{\varepsilon} + 25 + \frac{15}{4}\rho - \frac{35}{24}\pi^2 - \frac{1}{12}\rho\pi^2 - \frac{21}{2}\zeta_3 - \frac{17}{90}\pi^4 \\ &\quad + i\pi \left[\left(\frac{5}{8} - \frac{\pi^2}{6} \right) \frac{1}{\varepsilon} + \frac{35}{8} + \frac{3}{8}\rho - \frac{\pi^2}{4} - 7\zeta_3 \right] + \mathcal{O}(\varepsilon). \end{aligned}$$

Outlook

Extension of the subtraction method to NNLO.

Already done:

- Construction of the subtraction terms
→ Subtracted matrix elements can be integrated numerically !
- Integration of the subtraction terms for one-loop amplitudes with one unresolved parton.

To be done:

- Analytic integration over the unresolved phase space for double unresolved terms.