Progress in Lattice Field Theories

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Introduction

The only method known to treat field theories non-perturbatively without approximations is *Lattice Field Theory*

$$\mathcal{L}_{QCD} = -\frac{1}{2g_0^2}\,Tr\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \bar{\Psi}_f\{D + m_f\}\Psi_f$$

Discretize this system in a space-time lattice preserving:

- gauge invariance
- locality
- unitarity

Chiral symmetry for $m_f \longrightarrow 0$ is hard to implement (Nielsen-Ninomiya Theorem):

- Wilson fermions : $\sum_f \bar{\Psi}_f\{D + m_0 + aD^2\}\Psi_f$
- Staggered fermions : maintain a chiral $U(1)$ but break flavour symmetry

Becomes a finite calculation...

$$\langle \mathcal{O} \rangle = \frac{\int D[A_\mu] \; \mathcal{O} \; \det(D + m_f) \; e^{-S_{gauge}}}{\int D[A_\mu] \; \det(D + m_f) e^{-S_{gauge}}}$$

but $(T/a) \times (L/a)^3 \times 8 \times 4$ integrals! Ex: $a = 0.1$fm, $L = 2$fm, $L/a = 20$

The calculation requires statistical methods $\longrightarrow$ Monte Carlo approach

Universality warranties that, once we properly renormalize and take the $a \longrightarrow 0$ limit, we obtain QCD but...

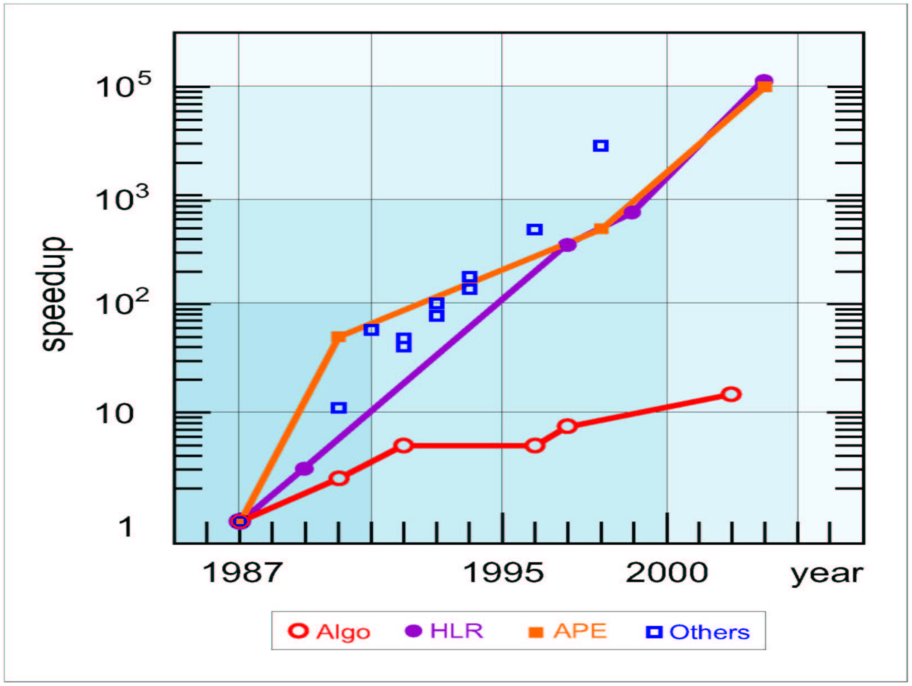
- Wilson fermions: fine tunning to flow to the chiral continuum theory
- Staggered fermions: $N_f = 4n$
  Getting rid of the additional flavours requires modifications that do not obviously preserve locality

$$\det(D + m_f) \longrightarrow \left[\det(D_{staggered} + m_f)\right]^{N_f/4}$$

Quenching remains an embarrasing but necessary approximation: neglect fermion loops

$$\det(D + m_f) \longrightarrow 1$$

In spite of the fact that the progress in this field would not have been possible without the exponential growth of the computer speed:



K.Jansen, Lattice Forum

Quenched QCD can be simulated in small PC clusters affordable to smaller groups!

There is no *iff* : the brute force method does not take you very far

Other aspects are equally essential:

- Numerical algorithms : can get you orders of magnitude!
- Improved actions

All lattice actions that preserve the symmetries of the continuum theory and have the same particle content lead to the same continuum limit: *Universality*

However you can speedup the approach to this limit by choosing a better action

- Asking the right questions

The lattice allows us to do experiments that we cannot do in the laboratory. We can setup the degrees of freedom in QCD in different (unphysical) conditions that are useful to prove different properties of the system:

$$V, \quad m \neq m_{phys}, \quad \nu$$

I will illustrate recent progress in the field by showing some selected examples of all these

Disclaimer:

- I cannot cover all the interesting results in this field: $\mathcal{O}(250)$ contributions to Lattice2003
- Mostly concentrated in QCD. Not cover promising new ideas to deal with SUSY theories

D.Kaplan *et al*, JHEP 0305 (2003);hep-lat/0302017;hep-lat/0307012

- I will not provide new world averages for CKM fits:

$$F_{B_d/B_s}, B_{B_d/B_s}, \xi, B_K$$

See:
EPS Parallel talks:
Heavy flavour physics by J. Heitger, C. Tarantino and H. Wittig
Light quark physics by L. Giusti
EPS Plenary Talk by S. Stone
CKM-Lattice Working Group:`http://www.cpt.univ-mrs.fr/ldg`
CERN CKM report: hep-ph/0304132

- For more technical accounts on these and other topics:
Plenary sessions at Lattice 2003: `http://www.rccp.tsukuba.ac.jp/lat03/`

Algorithmic improvements

I. Confinement and the QCD string:

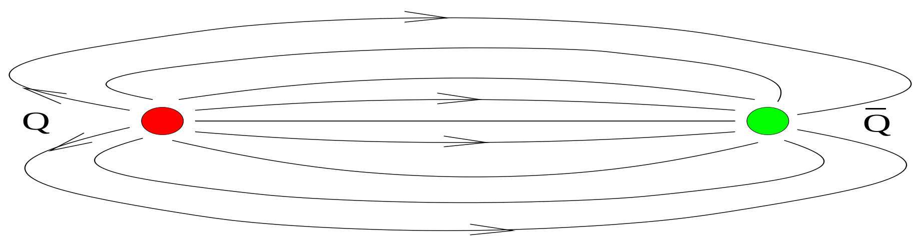
The linear growth of the potential between an infinitely heavy (static) quark and antiquark

$$\lim_{r\to\infty} V(r) = \sigma r \qquad \sigma : \text{string} \quad \text{tension}$$

Both this feature and the fact that resonances fall in Regge trajectories:

$$J_i \sim m_i^2$$

gave rise to the idea that non-abelian Yang-Mills theories could be (effective) string theories



In its less ambitious formulation, this correspondence implies that the low energy degrees of freedom are the "stringy" excitations of the thin flux tube that forms linking color charges:

$$S_{\text{eff}} = \int\int dz_0 dz_1 \, \tfrac{1}{2}\partial_a h \partial_a h + \ldots$$

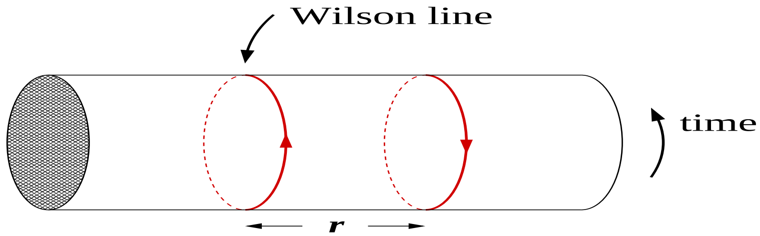
There is a universal prediction of this picture: $V(r) = \sigma r + \mu - \frac{\pi(D-2)}{24r} + \mathcal{O}\left(\frac{1}{r^2}\right)$

The Lüscher term is the same for all the string theories in the same universality class (depends on the number of bosonic and fermionic degrees of freedom)

Algorithmic challenge

On the lattice the static potential is measured from the correlation function of two large Wilson loops or Polyakov loops:

$$\lim_{T\to\infty} \langle P(r)P^*(0)\rangle = e^{-TV(r)}\left\{1 + \mathcal{O}\left(e^{-T\epsilon}\right)\right\} \qquad P(r) \equiv \mathrm{Tr}\left[U_0(\vec{x},a)U_0(\vec{x},2a)....U_0(\vec{x},Ta)\right]$$



This is a difficult measurement because signal/noise decreases exponentially with the loop size:

$$\langle P(r)P^*(0)\rangle \sim e^{-\sigma r T}$$

while the noise is approximately constant

Ex: increasing $\Delta A \sim 1\mathrm{fm}^2$, implies that the statistics has to increase by 3×10^4 for constant signal/noise!

Only in simple gauge theories such as 3d Z(2), the Lüscher term had been computed reliably
Caselle, *et al* 1996

Recent achievement: algorithm for any gauge theory Lüscher, Weisz JHEP0109(2001),0207(2002)

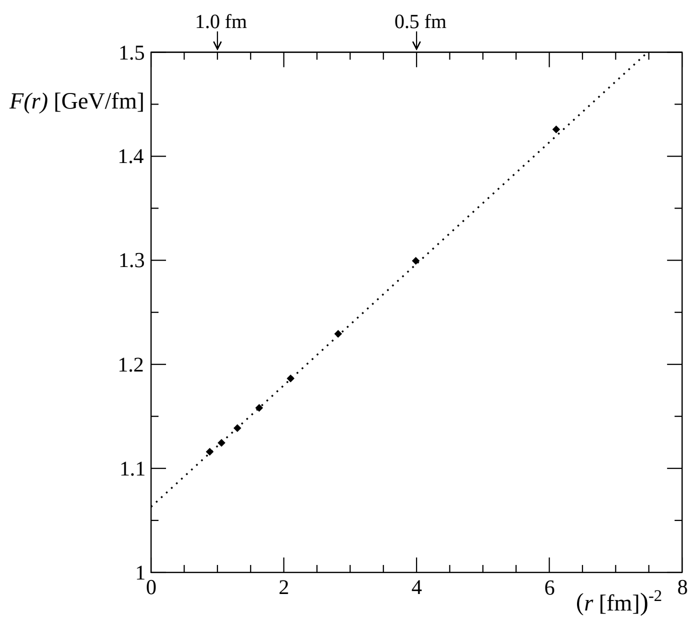
Using the locality of the action:

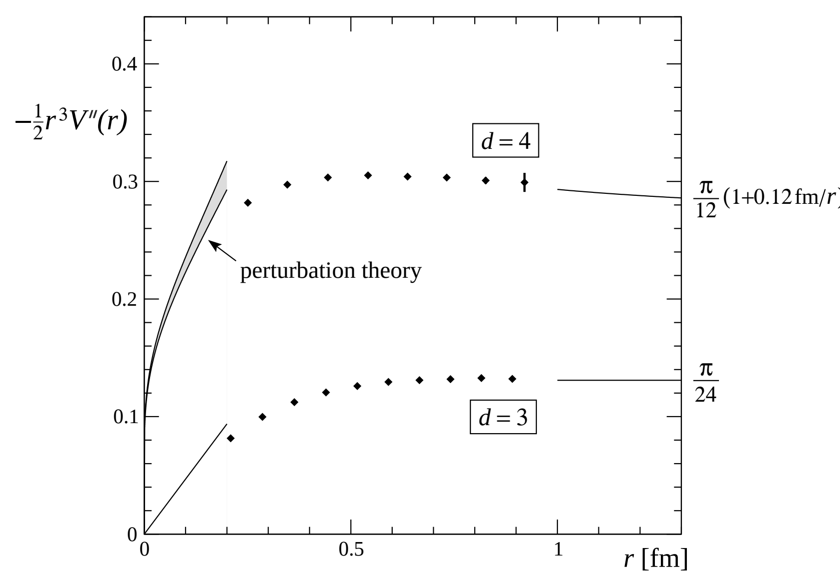
$$\langle \text{Product of links} \rangle = \text{Product}(\langle \text{Product of links} \rangle_{sublattice})$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$n \sim e^{\alpha r T} \qquad\qquad\qquad\qquad n \sim e^{\alpha r}$$

At least $O(10-100)$ more precise than previous calculations: $V'(r) = \sigma + \frac{\pi(D-2)}{24r^2} + \ldots$

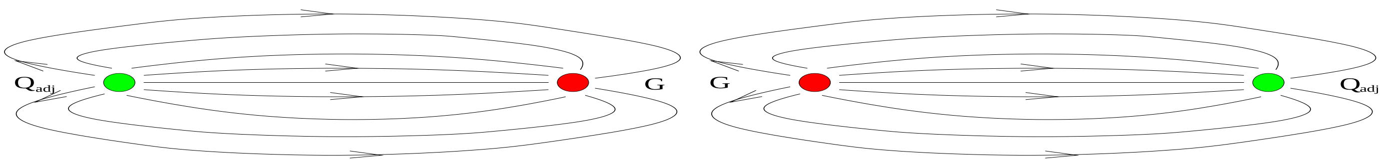


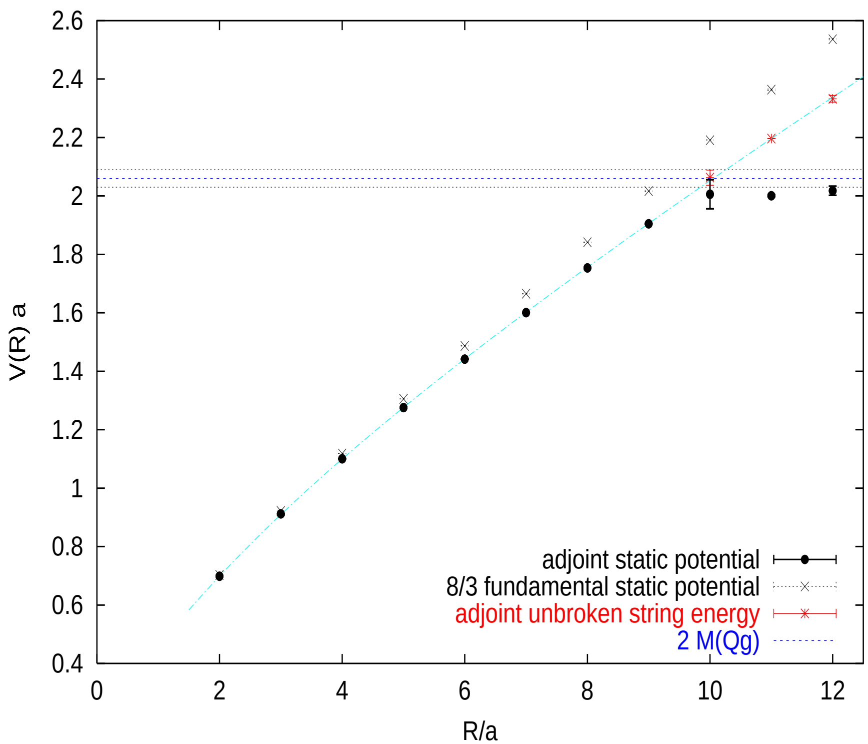


This is completely consistent with the *effective* string picture of QCD, and it is a strong constrain to the dreams of a fundamental equivalence between Yang-Mills and string theory!

String breaking from Wilson loops

If you consider the static potential between adjoint charges, the string is expected to break because adjoint charges can be screened by gluons:



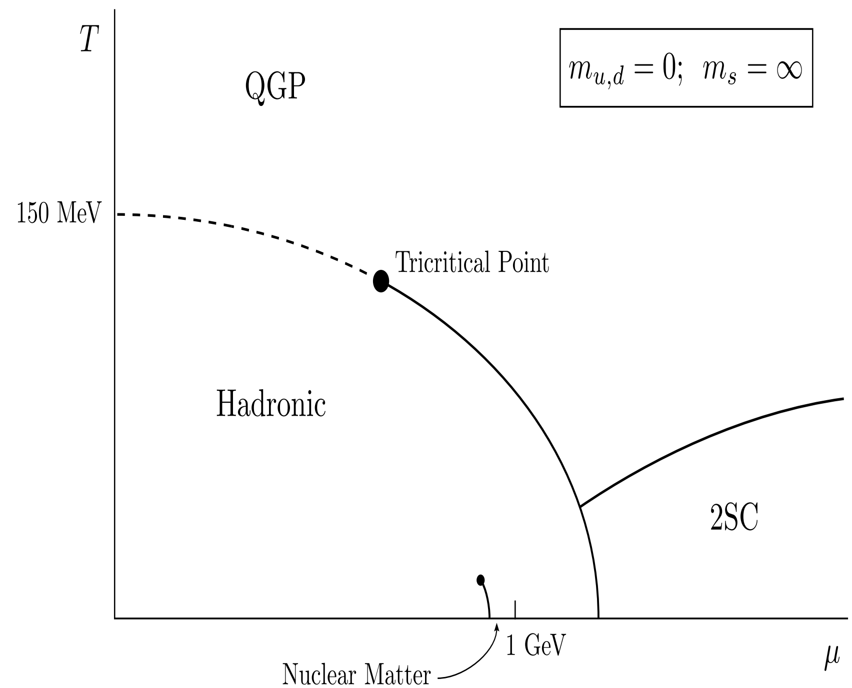


$SU(2)$ in 3D

$T \sim$ 2fm to observe string breaking!

Kratochvila,de Forcrand, hep-lat/0306011

II. QCD at finite density:



Rajagopal 1999

Algorithmic challenge

Infamous **sign** problem: the fermion determinant at finite μ is not positive definite and Monte Carlo methods fail

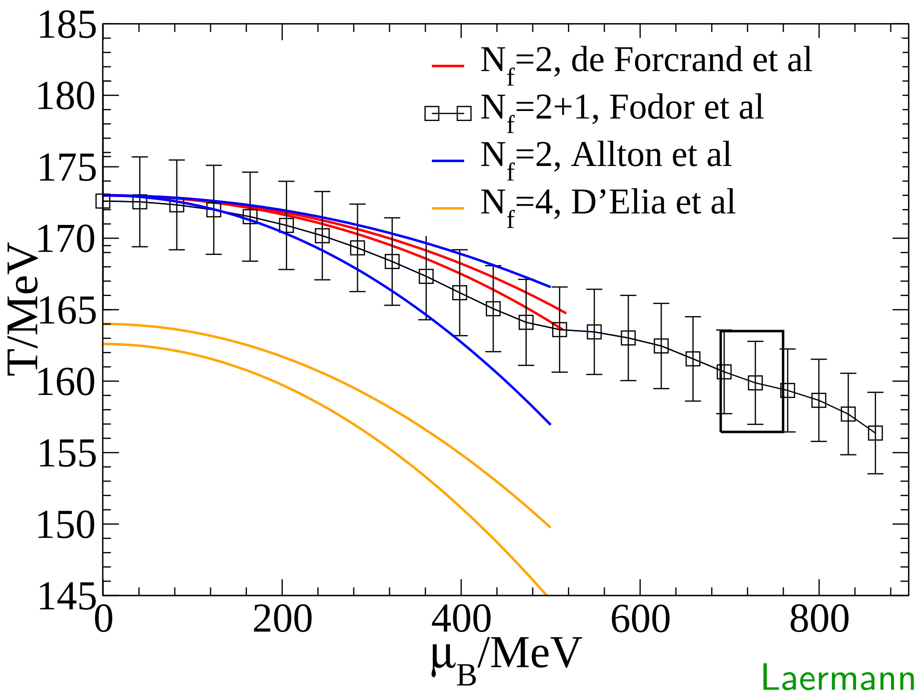
$$Z[\mu, T] = \int DA_\mu \underbrace{\det M(\mu, T)\; e^{-S_{gauge}(T)}}_{\text{not a probability!}}$$

Recent achievements New methods to explore $\mu_B \leq 500\mathrm{MeV}$ (RHIC, $\mu_B \leq 50$)

- Multiparameter reweighting Fodor, Katz PLB534(2002)

$$Z[\mu, T] = \int DU \underbrace{\det M(0, T_0)\, e^{-Sg[U,T_0]}}_{\text{importance sampling}} \underbrace{\frac{\det M(\mu,T)\ e^{-Sg[U,T]}}{\det M(0,T_0) e^{-Sg[U,T_0]}}}_{\text{observable}} = \left\langle \frac{\det(M(\mu,T))\ e^{-Sg[U,T]}}{\det(M(0,T_0))\ e^{-Sg[U,T_0]}} \right\rangle_{\mu=0,T_0}$$

- Taylor expansion in μ/T: much faster! Allton*et al* PRD66(2002)
- Analytic continuation $i\mu \to \mu$ through Taylor expansion de Forcrand, Philipsen NPB642 (2002)



Good agreement for $\mu_B < 500\mathrm{MeV}$

Location of the tricritical point (?)

Laermann, Philipsen hep-lat/0303042

Improved actions

Symanzik improvement:

Cutoff effects at finite a can be acounted for by the most general Lagrangian including higher order operators compatible with the lattice symmetries:

$$S_{\rm eff} = \int d^4x \left\{ \mathcal{L}_0^{d=4}(x) + a\mathcal{L}_1^{d=5}(x) + a^2\mathcal{L}_2^{d=6}(x) + \ldots \right\}$$

Wilson fermions: $\mathcal{L}_1 = c_1 \bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi$ $\longrightarrow$ scaling violations of $\mathcal{O}(a)$
Staggered fermions: $\mathcal{L}_1 = 0$ $\longrightarrow$ scaling violations of $\mathcal{O}(a^2)$

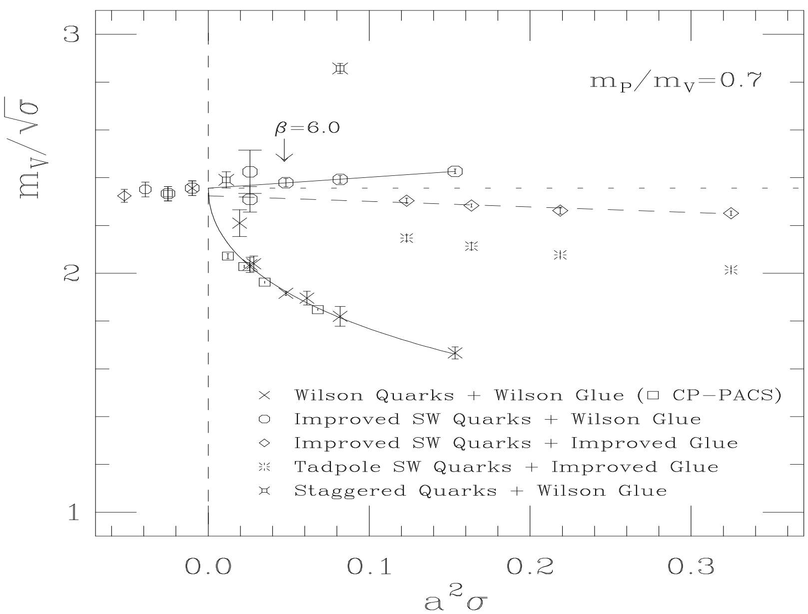
In all cases $\mathcal{L}_2$ contains many new couplings...

$$\mathrm{Observ}(a) = \mathrm{Observ}\left[1 + a\Lambda_1 + (a\Lambda_2)^2 + \ldots\right]$$

We can get rid of the term of $\mathcal{O}(a)$, by tuning the coupling $c_1 \longrightarrow 0$ non-perturbatively

Symanzik 1983; Sheikholeslami, Wohlert 1985; ALPHA coll. 1996

$\mathcal{O}(a)$ improvement has been implemented systematically for Wilson fermions for the action and quark bilinear operators



Sharpe ICHEP1998

Caveats:

- The size of the $\Lambda_i' s$ is not known a priori: a continuum extrapolation is still needed
- The $\Lambda_i' s$ can vary from observable to observable
- Partial improvements make extrapolations, when there is a sizeable change, more complicated

The improvement of the weak Hamiltonian (4-fermion operators in $\Delta B, \Delta S$ transitions) remains an enormous challenge $\longrightarrow$ exact chiral symmetry seems the only hope

Ginsparg-Wilson fermions

Lattice Dirac operators can be constructed which are local, do not suffer from the doubling problem and satisfy the Ginsparg–Wilson (GW) relation:

$$\{D, \gamma_5\} = aD\gamma_5 D \quad \leftrightarrow \quad \{D^{-1}_{xy}, \gamma_5\} = a\gamma_5\delta_{xy}$$

Ginsparg and Wilson PRD25 (1982)

From a RG blocking procedure from the continuum

$$e^{-\bar{\psi}D'\psi} = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\{-(\bar{\psi}-\bar{\phi})R(\psi-\phi)-\bar{\phi}D\phi\}$$

R is generically not chirally symmetric (e.g. proportional to unit matrix)

$$\gamma_5 D' + D'\gamma_5 = 2D'\gamma_5 R^{-1} D'$$

They realized that Ward identities associated with a standard chiral transformation are satisfied on shell at finite a:

$$lim_{m\to 0}\langle \delta^x_\chi S_f \; O(y)\rangle = 0, \quad |x-y| \neq 0$$

No explicit construction was found then and it fall in oblivion until P. Hasenfratz rediscovered it in 1997 and realized that his fixed-point Dirac operator satisfies it

$$(D^{FP})' = D^{FP}$$

P. Hasenfratz, NPB Proc. Suppl. (1998)

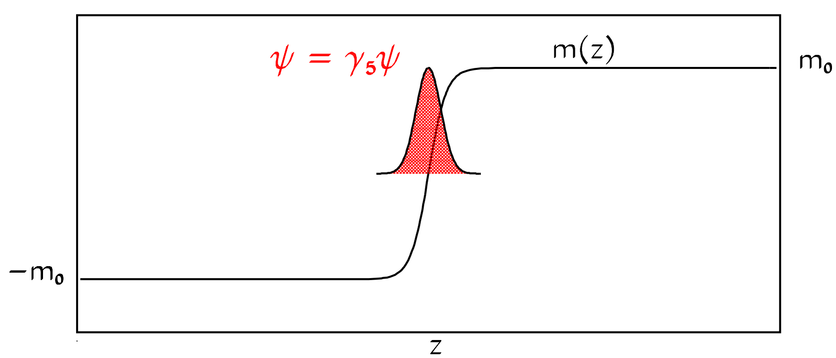
But the fixed-point Dirac operator is not an explicit construction either and truncation were needed...

Domain-wall in 5 dimensions

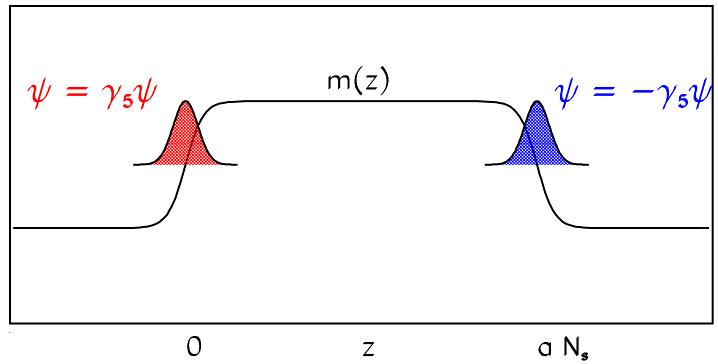
An infinite domain-wall (DW) in 5D leads naturally to chiral fermions in 4D

Rubakov, Shaposhnikov, PLB125 (1983)

Callan, Harvey, NPB250(1985)



On the lattice the DW construction a , a_s, N_s :



Kaplan, PLB288 (1992);
Shamir, NPB406 (1993);
Narayanan, Neuberger, NPB412 (1994)

$N_s \longrightarrow \infty$ at finite a and a_s , we expect two lattice "chiral" fermions $\longrightarrow$ satisfies GW relation !

The effective action and propagator of the light boundary fields can be described in terms of 4D operator aD_{N_s} which satisfies GW in $N_s \longrightarrow \infty$:

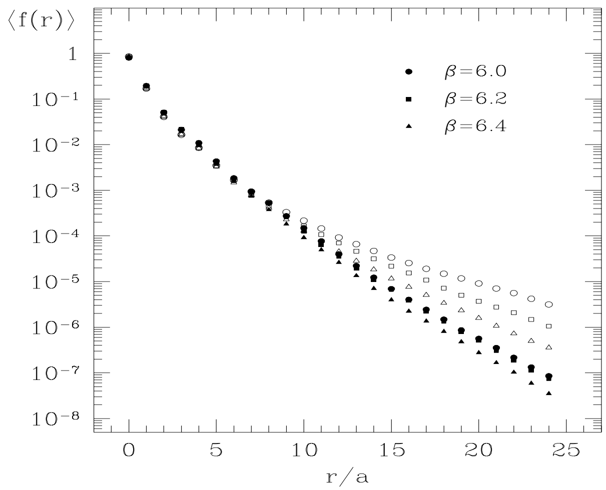
$$\lim_{a_s \to 0, N_s \to \infty} aD_{N_s} = aD_{ov} = 1 - \gamma_5 \mathrm{sign}(\mathrm{Q}) \qquad \mathrm{Q} \equiv \gamma_5(\mathrm{m_0} - \mathrm{D_W})$$

Neuberger, PLB417 (1998)

D_{ov} was the first explicit construction of GW fermions: $\{D_{ov}^{-1}, \gamma_5\} = a\gamma_5$

Neuberger, PLB427 (1998)

- Has the right continuum limit
- No doublers
- In spite of its looks, it is a local operator:



$$||D_{ov}(0,r)|| \leq e^{-\gamma|r|/a}$$

H., Jansen, Lüscher NPB552 (1999)

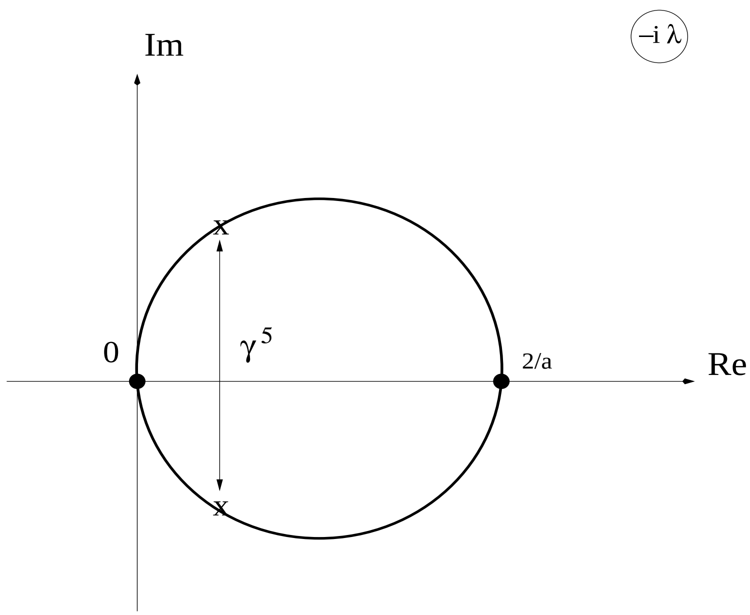
The GW relation implies an exact symmetry :

$$\delta_\chi \Psi = \epsilon\, \gamma_5 (1 - aD)\Psi \qquad \delta_\chi \bar{\Psi} = \epsilon \bar{\Psi}\gamma_5 \quad \rightarrow \delta_\chi S_f = 0$$

Lüscher PLB428 (1998)

$U_A(1)$ anomaly is recovered due to the non-invariance of the fermion measure under a singlet chiral rotation:

$$\langle \delta_\chi O \rangle_F = \mathrm{Tr}\left[\gamma_5(1 - aD/2)\right] \langle O \rangle_F$$



$$\mathrm{Tr}[\gamma_5(1 - aD/2)] = N_f \times index(D)$$

A GW operator satisfies an exact index theorem and topological sectors can be distinguished!

Hasenfratz, Laliena, Niedermayer, PLB427 (1998)

Flavour symmetry is exact:

$$\Psi_{R,L} = \hat{P}_{\pm}\Psi \qquad \bar{\Psi}_{L,R} = \bar{\Psi} P_{\pm}$$

with $P_{\pm} \equiv (1 \pm \gamma_5)/2,\ \hat{P}_{\pm} \equiv (1 \pm \gamma_5(1 - aD))/2$

$$\bar{\Psi} D \Psi = \bar{\Psi}_L D \Psi_L + \bar{\Psi}_R D \Psi_R,$$

There is an exact $SU(N_f)_R \times SU(N_f)_R$ symmetry:

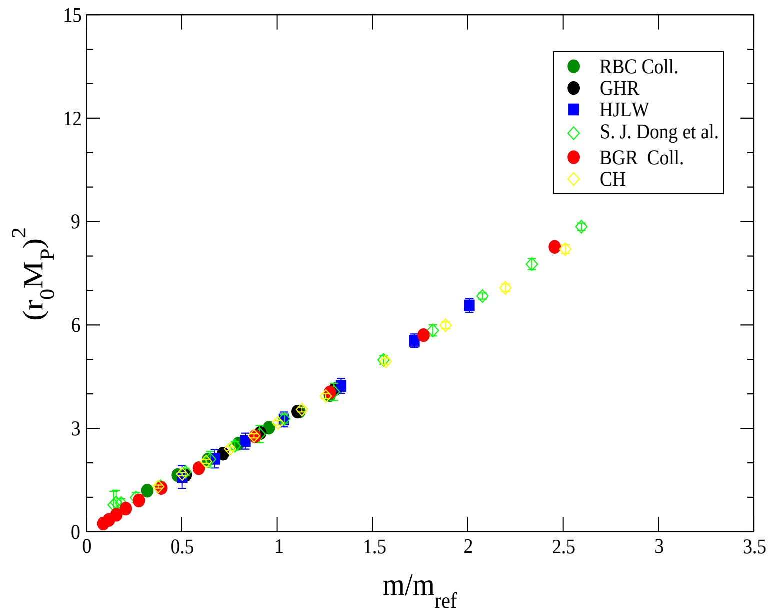
$$\Psi_L \longrightarrow V_L \Psi_L \qquad \Psi_R \longrightarrow V_R \Psi_R \quad V_{L,R} \in SU(N_f)_{L,R}$$

Adding quark masses: $\bar{\Psi}_L m \Psi_R + \bar{\Psi}_R m^{\dagger} \Psi_L$

- There is a conserved axial current in the chiral limit $m \longrightarrow 0$
- Operator classification and mixing is enormously simplified: four fermion operators only mix with those in the same chiral representation
- Scaling violations are $O(a^2)$: the exact chiral symmetry forbids all operators of $d = 5$ in the action

GW are expensive $\mathcal{O}(100)$ but feasible in the quenched approximation!

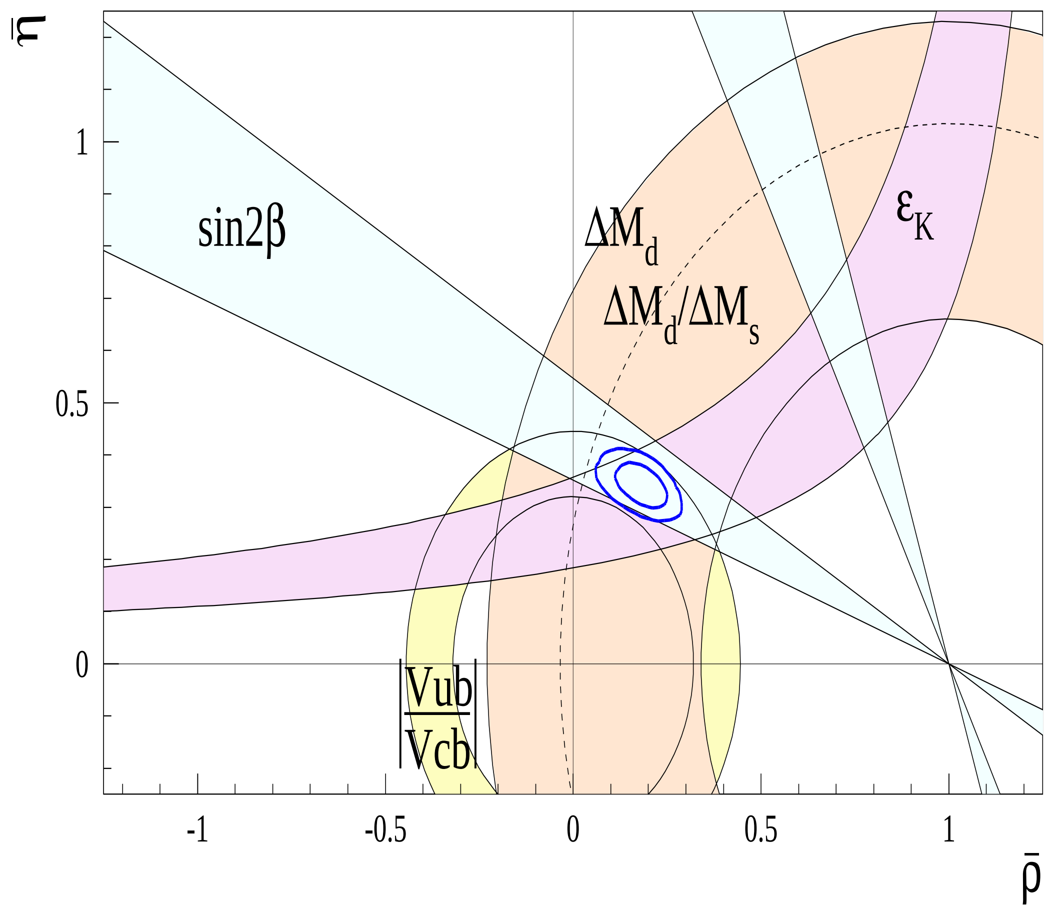
$$L = 1 - 3 \text{ fm}, \; a = 0.08 - 0.2 \text{ fm}$$



Review by Giusti, hep-lat/0211009

Two recent applications:

- B_K
- QCD versus Random Matrix Theory (RMT)



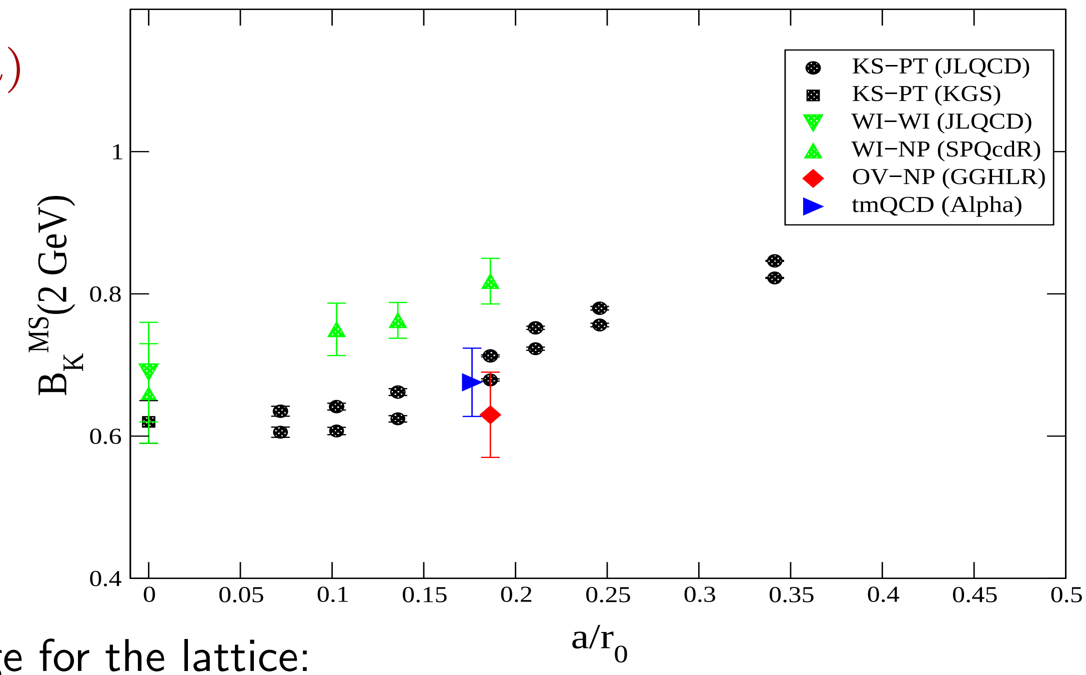
Ciuchini, *et al*, hep-ph/0307195

$$\mathcal{H}_{\Delta S=2} = C_W(\mu)\mathcal{O}_{\Delta S=2}(\mu) + \ldots$$

$$\langle \bar{K}^0 | \mathcal{O}_{\Delta S=2}(\mu) | K^0 \rangle = \frac{8}{3} M_K^2 F_K^2 B_K(\mu)$$

$$\mathcal{O}^{bare}_{\Delta S=2} = [\bar{s}(V-A)d][\bar{s}(V-A)d]$$

B_K from Quenched QCD



See Giusti's parallel talk

$K_0 - \bar{K}_0$ mixing has been a big challenge for the lattice:

- Wilson fermions: mixing with other chiral structures $V \times V - A \times A$, $S \times S \pm P \times P$, $T \times T$...Recently smart tricks to get rid of these mixings

  Frezzotti *et al* JHEP0108(2001); Becirevic *et al*, PLB487(2000); Guagnelli *et al* NPPS 106 (2002)

- Staggered fermions: mixing with other flavour breaking structures. Renormalization done perturbatively

With GW only multiplicative renormalization:

Non-perturbative renormalization (vs. staggered)
Scaling violations of $O(a^2)$ (vs. Wilson)
} Competitive!

QCD and RMT

Iff there χSB, the QCD partition function at fixed topological charge is conjectured to be equivalent to that of a Random Matrix Theory: Shuryak, Verbaarschot 1993

$$\lim_{N\to\infty, m\rho(0)N=fixed} Z_\nu^{RMT}(m) = \lim_{V\to\infty, m\Sigma V=fixed} Z_\nu(m)$$

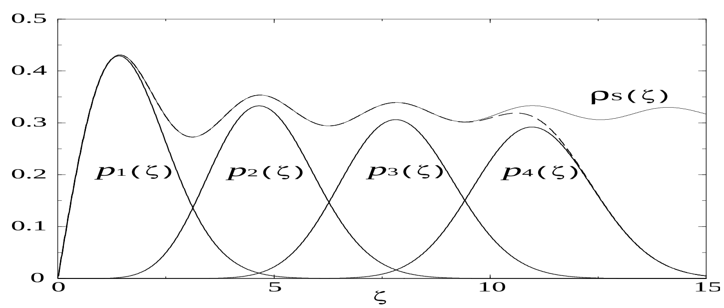
$$Z_\nu^{RMT} \equiv \int dW \prod_{f=1}^{N_f} \det(iM+m) \exp\left[-N/2\mathrm{Tr}V(M^2)\right],$$

$M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix}$, where W are random complex matrices of rectangular size $N \times (N+\nu)$

If $\rho(0) \neq 0$, spectrum close to zero in the microscopic limit is universal (independent of $V(M)$)

$$\rho_\nu(\lambda) = \frac{\Sigma\zeta}{2}\left(J_{N_f+\nu}(\zeta)^2 - J_{N_f+\nu-1}(\zeta)J_{N_f+\nu+1}(\zeta)\right) \quad \zeta \equiv \lambda\Sigma V$$

Verbaarschot, Zahed, PRL70(1993)

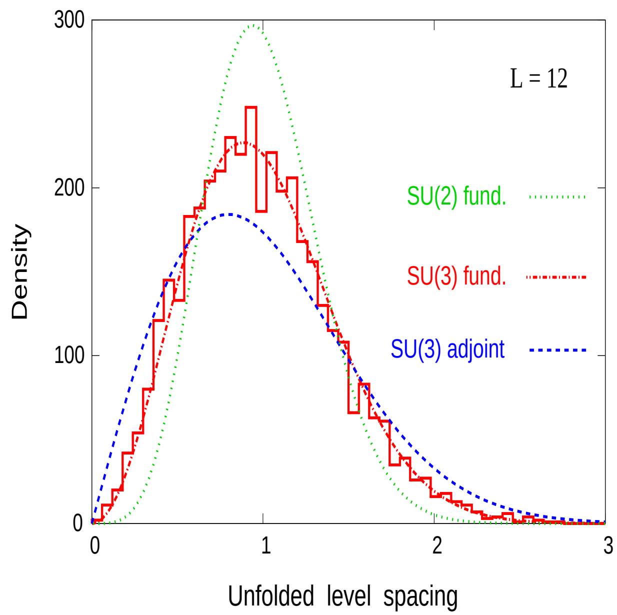


Chiral symmetry is essential to test this conjeture using the lattice

Previous simulations with staggered (not in the same universality class at finite a) and GW fermions very far from the continuum limit

Review by P.H. Damgaard, hep-lat/0110192

$a = 0.09$ fm, $L = 1.1$ fm



Three Universality classes:

Complex: $SU(N_f) \times SU(N_f) \rightarrow SU(N_f)$ ChUE

Pseudo-real: $SU(2N_f) \rightarrow Sp(2N_f)$ ChOE

Real: $SU(2N_f) \rightarrow SO(2N_f)$ ChSE

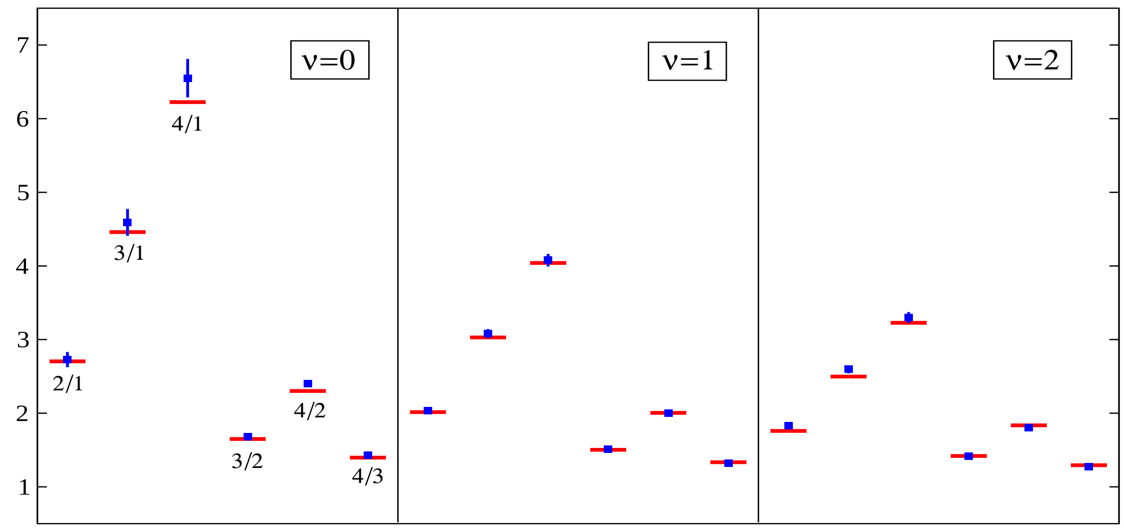
Bietenholz, Jansen, Shcheredin hep-lat/0306022

Optimized algorithms for overlap fermions (Giusti *et al.* hep-lat/0212012) have made possible to test RMT at larger volumes

The agreement of the ratios of the averages of individual eigenvalues is truly remarkable!

$a = 0.09$ fm, $L = 1.5$ fm

$\frac{\langle \lambda_i \rangle_\nu}{\langle \lambda_j \rangle_\nu}$

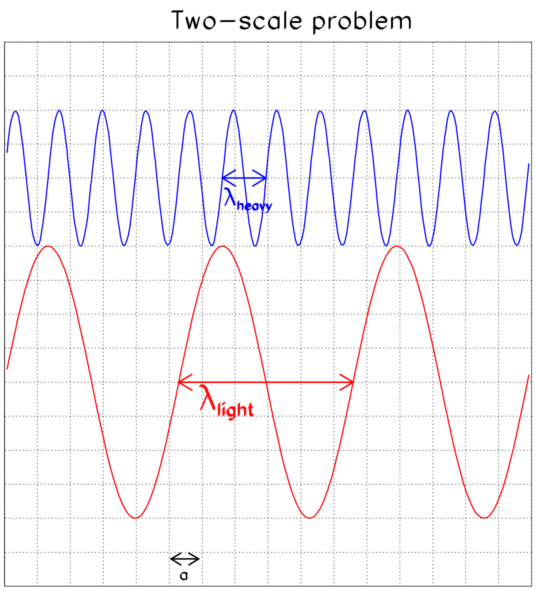


Giusti, Lüscher, Weisz, Wittig in progress

- Similar results at smaller $a = 0.07$fm
- At smaller $L \sim 1.2$ fm deviations from RMT are clearly measured
- This will probably provide the most precise determination of Σ

Ask the right questions

A typical problem in lattice QCD is the two or multi-scale problem:



$\lambda_{\text{light}} \gg \lambda_{\text{heavy}}$

The requirements are:

$\lambda_{\text{heavy}}/a \gg 1 \quad \longrightarrow$ small cutoff effects
$\lambda_{\text{light}}/L \ll 1 \quad \longrightarrow$ small finite size effects

L/a has to be increased by a factor $\sim \frac{\lambda_{\text{light}}}{\lambda_{\text{heavy}}}$ with respect to the one-scale problem

This situation is present in:

- Heavy quark physics: $\lambda_{\text{light}} \sim \Lambda_{QCD}^{-1} \gg \lambda_{\text{heavy}} \sim m_b^{-1}$
- Light quark physics: $\lambda_{\text{light}} \sim m_\pi^{-1} \gg \lambda_{\text{heavy}} \sim \Lambda_{QCD}^{-1}$

Heavy Quark Physics

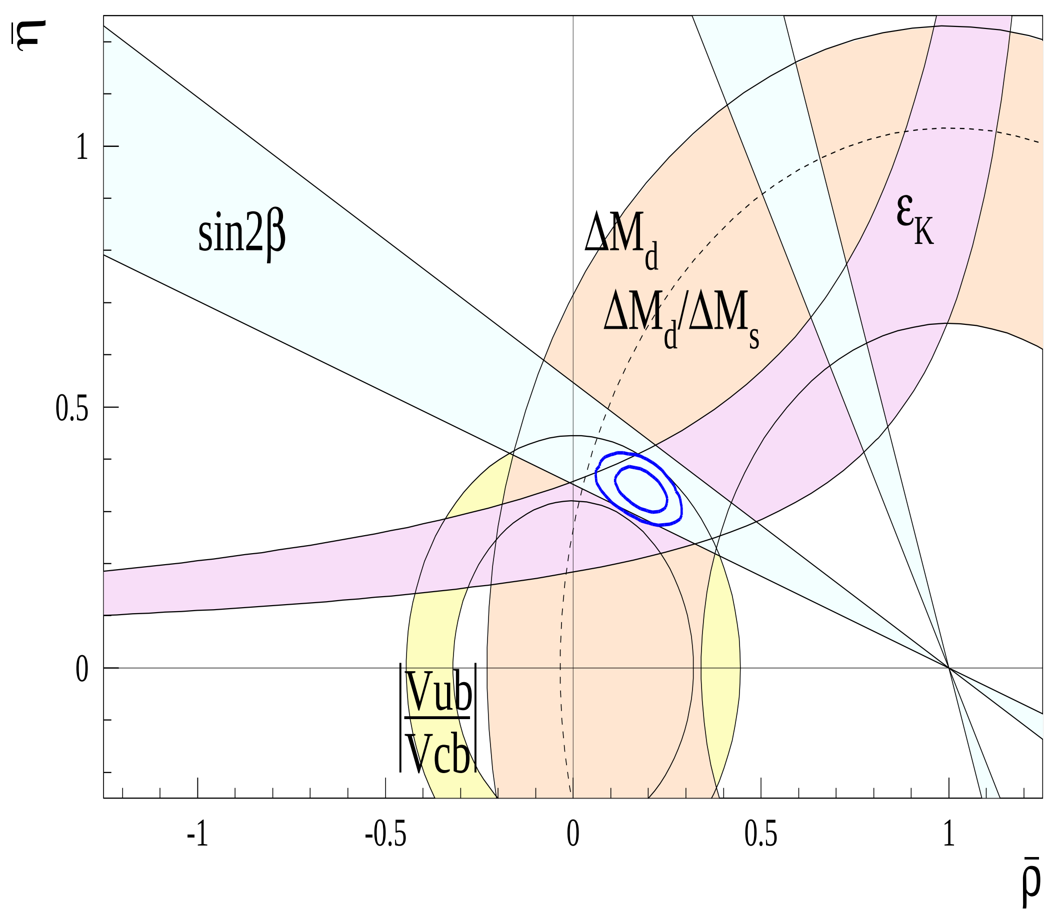
The lattice can in principle provide precise numbers for many quantities of phenomenological interest:

$$F_{B,B_s},\, B_{B,B_s},\, \xi \equiv \frac{F_{B_s}\sqrt{B_{B_s}}}{F_{B_d}\sqrt{B_{B_d}}}$$

$$\langle M|\mathcal{O}_{\Delta M=2}|M\rangle = \frac{4}{3} m_M^2 F_M^2 B_M \quad \langle B_d|\bar{b}\gamma_\mu\gamma_5 d|0\rangle = i p_\mu F_{B_d}$$

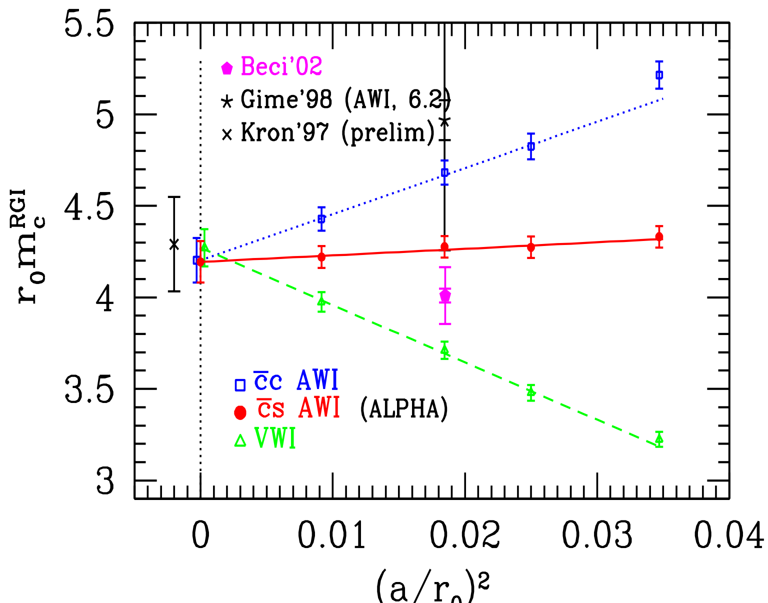
A lot of results have been obtained in recent years using a plethora of methods to overcome the two-scale problem

Unfortunately, errors are mostly dominated by systematics associated to the approximations: the uncertainty in systematic errors is large!

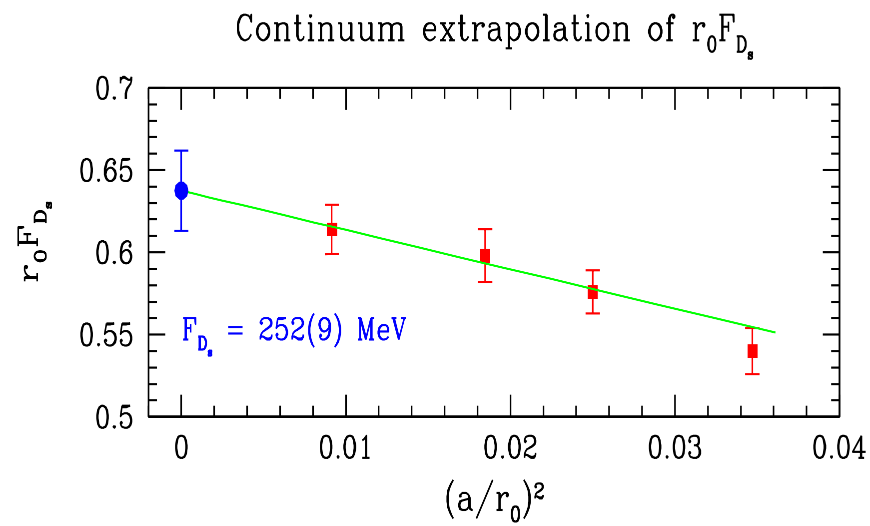


$\bar{\eta}$
1
0.5
0
sin2β
ΔM_d
$\Delta M_d/\Delta M_s$
ε_K
$\left|\frac{Vub}{Vcb}\right|$
-1
-0.5
0
0.5
1
$\bar{\rho}$

- **Class A**: relativistic heavy quarks
  Lattice sizes can accomodate reliably quarks in the charm region! With $\mathcal{O}(a)$-improved Wilson action cutoff effects are under control



Rolf, Sint JHEP 0212(2002)



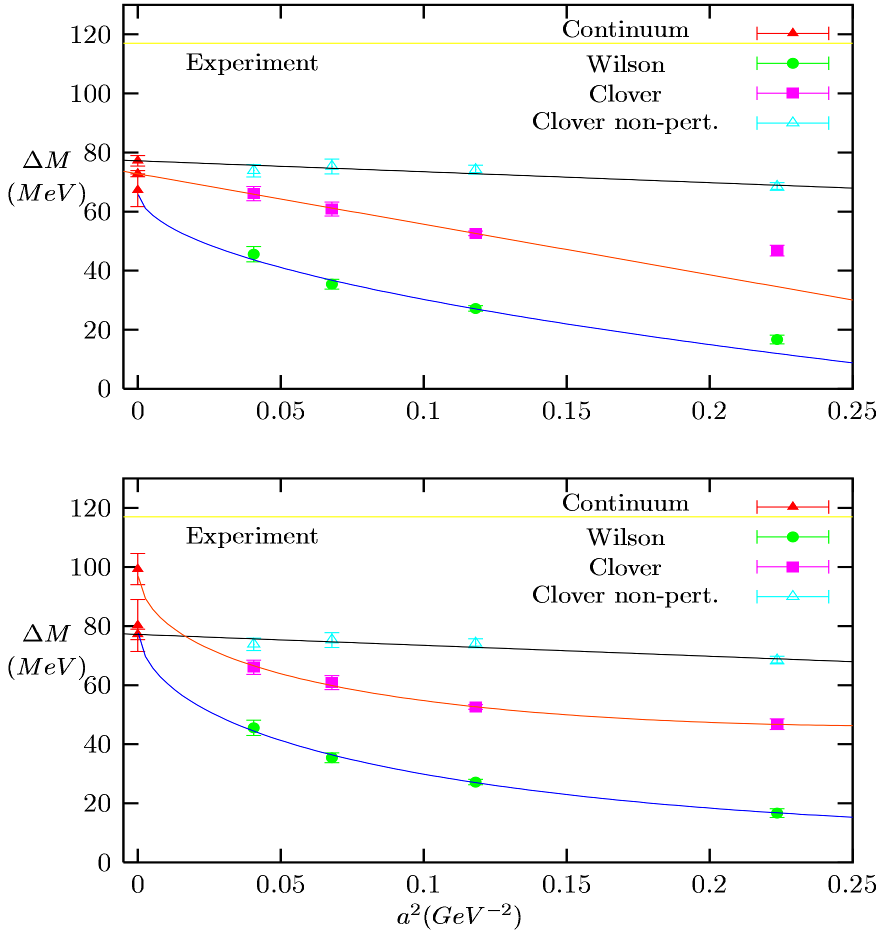
Jüttner, Rolf, PLB560(2003)

$$m_c^{\overline{MS}}(m_c^{\overline{MS}}) = 1.301(34) \qquad\qquad F_{D_s} = 252(9)\mathrm{MeV}$$

However, large extrapolations are needed $m_h \longrightarrow m_b$ and this is specially dangerous in the presence of large cutoff effects

$$\lim_{m_h \to m_b} \ \lim_{a m_h \to 0} \mathcal{O}(m_h, a m_h)$$

Hyperfine splitting in charmonium in the quenched approximation:



$$\Delta M = M(J/\psi) - M(\eta_c) = 77(2)(6)\text{MeV}$$

Experiment :117MeV

QCD-TARO coll. hep-lat/0307004

– Non-perturbative improvement is very important for reliable continuum extrapolations!
– Quenching effects are large for this quantity.

- **Class B**: non-relativistic expansion in $\frac{\Lambda_{QCD}}{m_b}$

$$\mathcal{L}_{HQET} = \bar{\Psi}_h D_0 \Psi_h - \frac{1}{2m_b}\bar{\Psi}_h \mathbf{D}^2 \Psi_h - \frac{c_\sigma}{2m_b}\bar{\Psi}_h B \cdot \sigma \Psi_h + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b}\right)^2$$

Use the effective theory on the lattice instead, since the cutoff can be much lower than m_b!

Matching and renormalization of the effective theory must be done non-perturbatively because there are generically power divergences:

$$\text{Matching at } l \text{ } loop \Rightarrow \Delta c_k \sim \frac{g_0^{2(l+1)}}{a} \sim \frac{1}{aln(a\Lambda)^{l+1}} \xrightarrow{a\to 0} \infty$$

The continuum limit cannot be taken!

Using HQET at NLO implies that we remain with uncertainties $\mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b}\right)^2$, but it is important:

- Combining the static limit with the relativistic approach allows an interpolation to m_b

- The non-perturbative determination of the couplings and matrix elements in HQET is very valuable information for continuum phenomenology

Recent progress: heavy quark physics in the quenched approximation can enter the no-systematic-error era with the use of finite-size scaling techniques

Class A: the b-quark can be simulated in a small volume $L_0 m_b \gg 1, m_b a \ll 1, L_0 \Lambda_{QCD} \leq 1$

Guagnelli *et al*, PLB546(2002); de Divitiis, *et al* hep-lat/0307005, 0305018

$$\mathcal{O}(m_h, m_l, L_0) \longrightarrow \text{large finite volume effects}$$

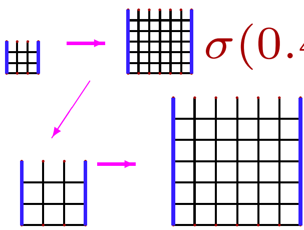
Define a step scaling function: $\sigma(m_h, m_l, L) \equiv \frac{\mathcal{O}(m_h, m_l, 2L)}{\mathcal{O}(m_h, m_l, L)}$

$$\mathcal{O}(m_h, m_l, 2^n L_0) = \prod_{i=1}^{n-1} \sigma(m_h, m_l, 2^i L_0)\; \mathcal{O}(m_h, m_l, L_0)$$

The main observation is that the step functions:

- are finite as $a \longrightarrow 0$
- are very smooth functions of m_h: constant up to $\mathcal{O}(1/m_h L)$ since $\mathcal{O}(m_l/m_h)$ cancel

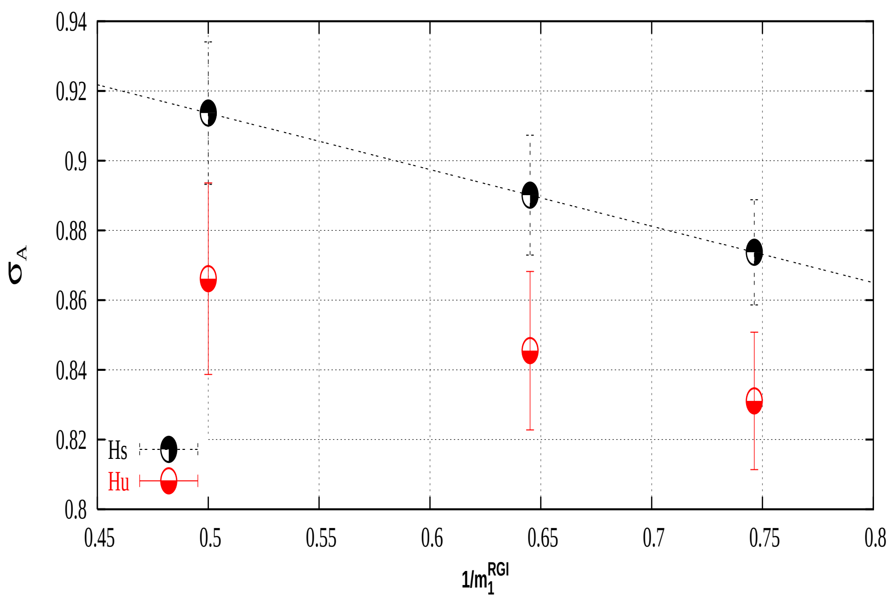
The series of step functions can be computed in a series of lattices with increasing a and L and roughly fixed am_h and m_hL:

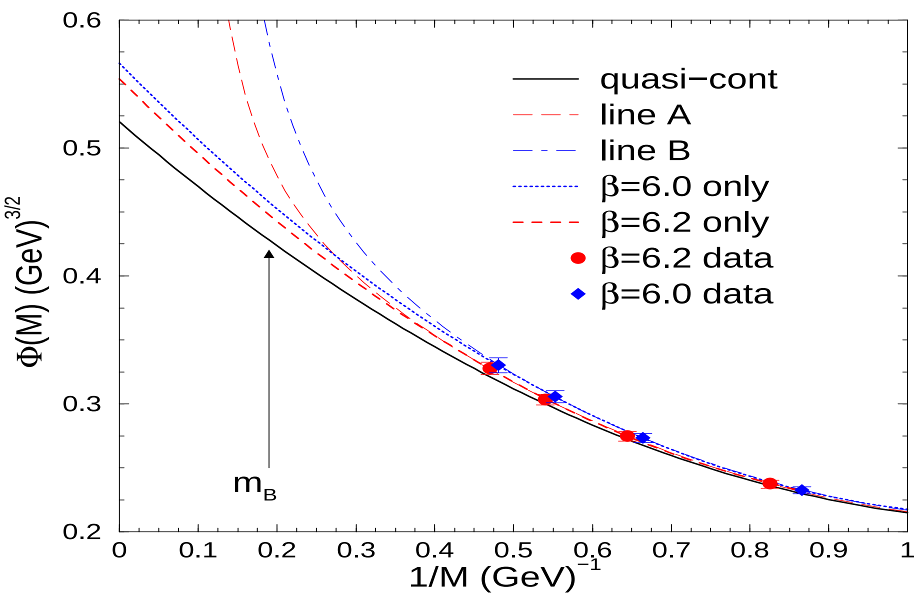


$\sigma(0.4\text{fm}),\ a < 0.05\text{fm} \to 0, m_h > m_b/2 \to m_b$

$\sigma(0.8\text{fm}),\ a < 0.1\text{fm} \to 0, m_h > m_b/4 \to m_b$

The extrapolation to m_b in the largest volume is a 6% effect as opposed to a 20% effect in the standard relativistic approach!

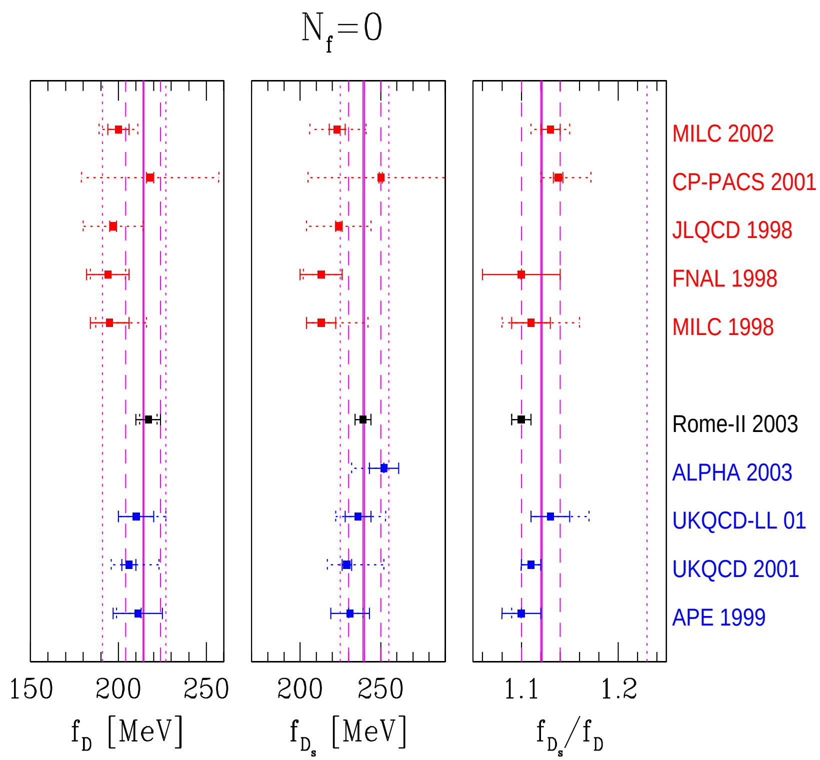


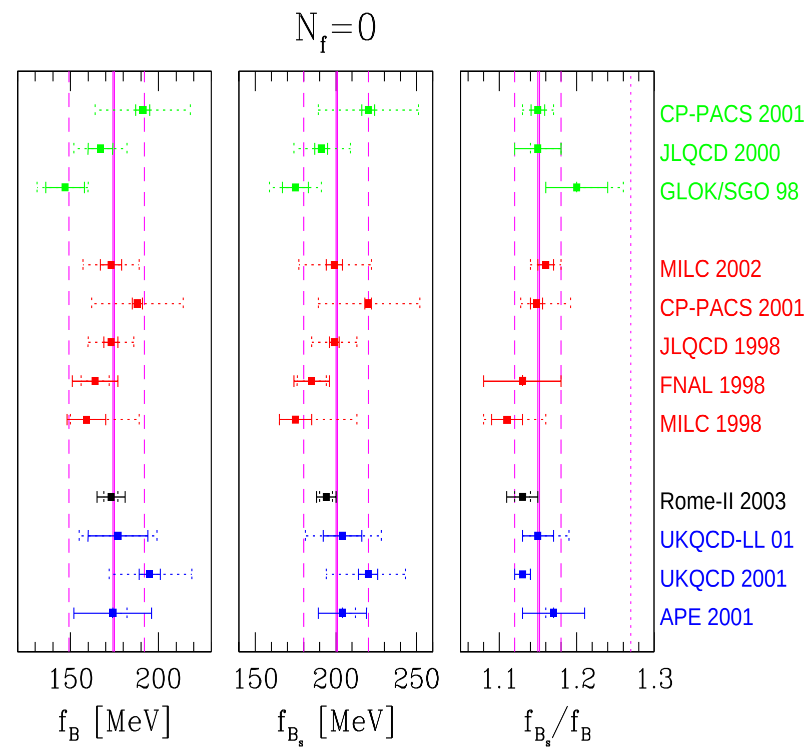


De Divitiis hep-lat/0307005

UKQCD coll., NPPS 106 (2002)

New results on f_{B,B_s}, f_{D,D_s}, M_b with significantly smaller systematic uncertainties (other than *quenching* !)





Wittig's parallel talk

Class B: non-perturbative matching and renormalization of HQET at finite volume

Heitger, Sommer, NPPS106(2002); Heitger, Kurth, Sommer, hep-lat/0302019; Della Morte *et al* hep-lat/0307021

$$\mathcal{O}^{QCD}(L, m_l, m_b) = \mathcal{O}^{HQET(n)}(L, m_l) + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b^{n+1}}\right)$$

The running to $L \to \infty$ is done using finite-size scaling in the effective theory and $a^{-1} \ll m_b$:

Example: b-quark mass in the static limit

$\Gamma(L)$ = energy of a state with the quantum numbers of a B in V= L^4

experiment | Lattice with $am_{\mathrm{q}} \ll 1$

$M_B = 5.4\,\mathrm{GeV}$ | $\Gamma_{QCD}(L_0, M)$

$\downarrow$ | $L_i = 2^i L_0$ | $\downarrow$

$\Gamma_{stat}(L_2)$ $\xleftarrow[\sigma(L_1)]{}$ $\Gamma_{stat}(L_1)$ $\xleftarrow[\sigma(L_0)]{}$ $\Gamma_{stat}(L_0)$

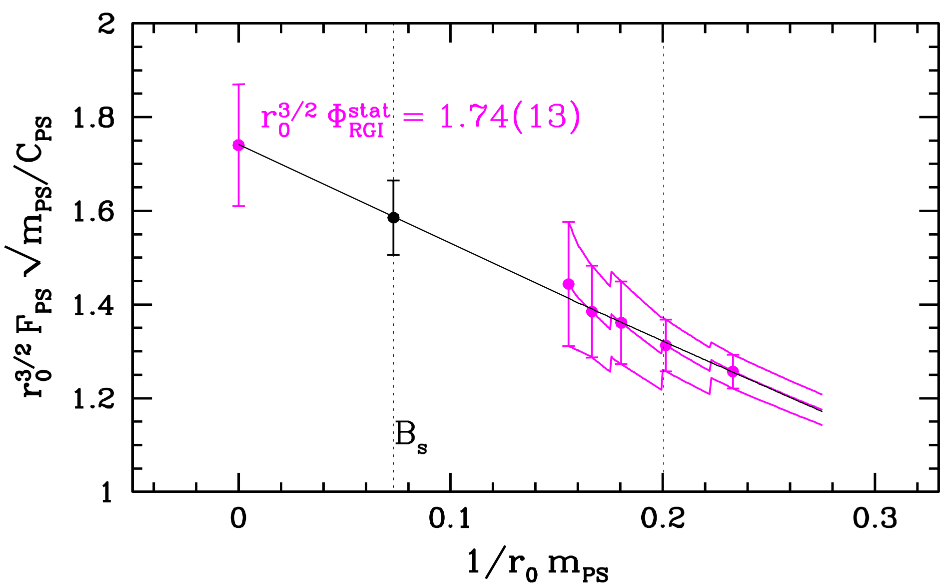
$$\underbrace{M_B}_{\text{exp.}} = \underbrace{\Delta\Gamma_{stat}(2^2 L_0, a)}_{a<0.07fm\to 0} + \underbrace{\Delta\Gamma_{stat}(2L_0, a)}_{a<0.05fm\to 0} + \underbrace{\Delta\Gamma_{stat}(L_0, a)}_{a<0.025fm\to 0} + \underbrace{\Gamma_{QCD}(L_0, m_b)}_{a<0.0125fm\to 0} \qquad L_0 = 0.2fm$$

with $\Delta\Gamma_{stat}(L, a) \equiv \Gamma_{stat}(2L, a) - \Gamma_{stat}(L, a)$

Results in the static limit for M_b and F_B in the static limit:

$$m_b^{\overline{MS}}(m_b^{\overline{MS}}) = 4.13(2)(4)\mathrm{GeV} + \mathcal{O}\left(\frac{\Lambda}{m_b}\right)$$

and more recently for F_{B_s} by interpolation between the relativistic and static result



Preliminary

$$F_{B_s} = 205(12)\mathrm{MeV}$$

Heitger's parallel talk

Same techniques can be applied to higher orders in $\mathcal{O}(1/m_b)$

These new ideas have not yet been applied to matrix elements

New lattice $\oplus$ HQET determination at NLO of lifetime ratios and width differences show better agreement with experiment:

NLO: M. Beneke *et al*; E. Franco *et al*; Dighe *et al*; Ciuchini *et al*

Matrix elements: Di Pierro, Sachrajda; Gimenez, Reyes; UKQCD; D. Decirevic *et al*; JLQCD

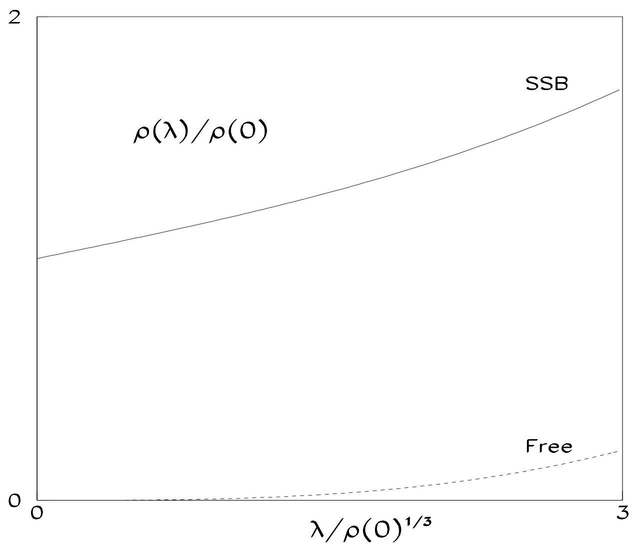
	LO	NLO	$Exp.$
$\frac{\tau(\Lambda_b)}{\tau(B_d)}$	$0.93(4)$	$0.88(5)$	$0.786(34)$
$\frac{\tau(B^+)}{\tau(B_d)}$	$1.01(3)$	$1.08(2)$	$1.085(17)$
$\frac{\tau(B_s)}{\tau(B_d)}$	$1.00(1)$	$1.00(1)$	$0.951(38)$
$\frac{\Delta\Gamma_d}{\Gamma_d}$		$0.0024(6)$	$0.008(37)(18)$
$\frac{\Delta\Gamma_s}{\Gamma_s}$		$0.074(24)$	$0.07^{(9)}_{(7)}$

See Tarantino's parallel talk

Light Quark Physics

The approach to the regime of the light quark masses is one of the most difficult problems in Lattice QCD both quenched and unquenched

$$m_q \geq \frac{m_s}{2} \quad m_{PS} \simeq M_K$$



χSB: $\lim_{m\to 0}\langle \bar{q}q\rangle = -\pi\rho(0) \neq 0$

$\lambda_{min} \sim \Delta\lambda \sim \frac{1}{\Sigma V}$

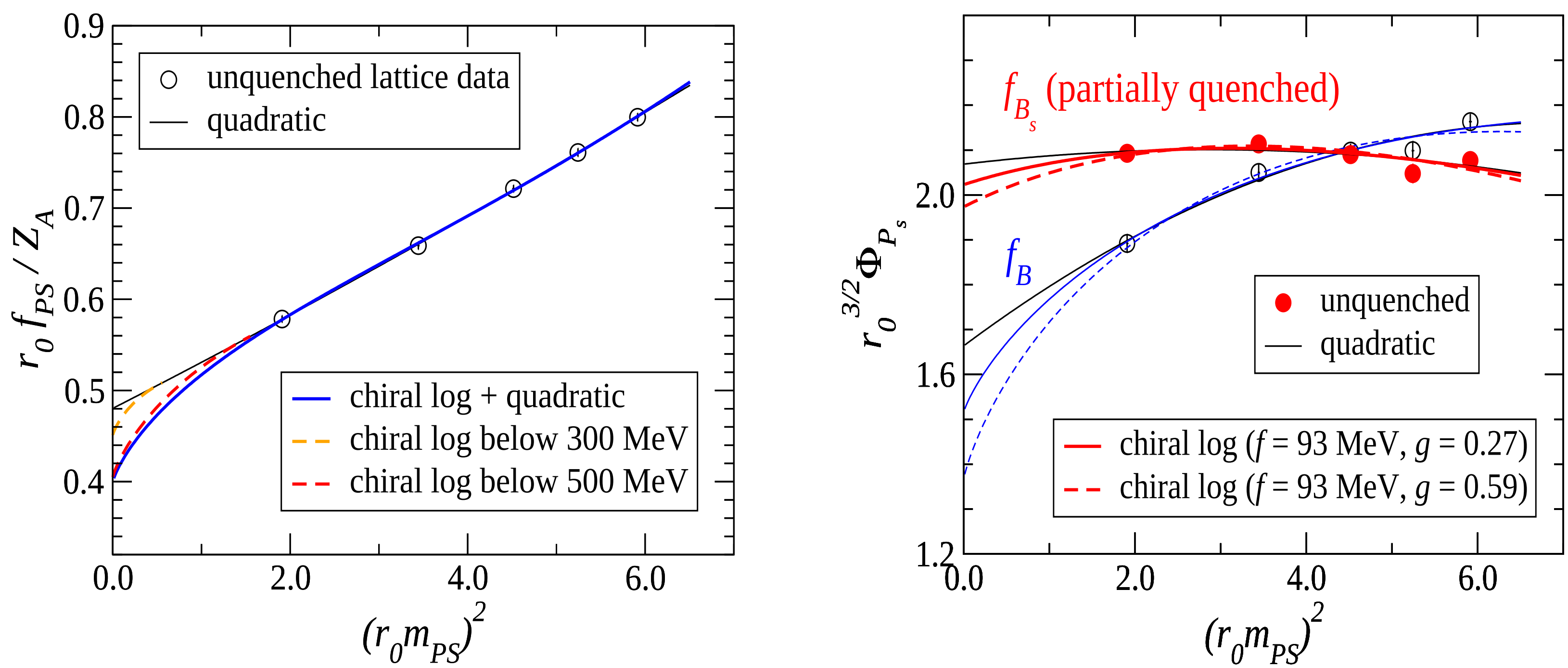
Algorithmic problem:

- Quenched case: cost$(D^{-1}) \sim 1/\lambda_{min}$ efficient algorithms for Ginsparg-Wilson fermions
- Unquenched case: cost $\sim 1/\lambda^3_{min}$!

Two-scale problem: $m_{PS} \ll \Lambda_{QCD} \quad L \gg m_{PS}^{-1}$

The standard approach: use χPT to extrapolate results at $m_q \simeq m_s/2 \to 0$

This is the source of one of the most important systematic uncertainties both in light (e.g. F_π) and heavy quark physics (e.g. B_{B_d}):



JLQCD coll.,hep-lat/0307039

Lattice $\oplus$ χPT

The interactions of the light mesons at low momenta are determined to a great extent by the pattern of chiral symmetry breaking

The QCD χ-Lagrangian incorporates automatically all the Ward identity relations and parametrizes what is not determined by symmetry by a set of low energy constants

Weinberg, Gasser and Leutwyler

$$\mathcal{L}_\chi^{QCD} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

$$\begin{aligned}
\mathcal{L}_\chi^{(2)} &= \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[e^{i\theta/N_f} M U + U^\dagger M^\dagger e^{-i\theta/N_f}\right] \\
\mathcal{L}_\chi^{(4)} &= L_1 \mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right]^2 + L_2 \left(\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\nu U\right]\right)^2 + \ldots
\end{aligned}$$

with

$M \longrightarrow$ quark mass matrix

$U = e^{i2\Phi/F}$ $\quad \Phi \longrightarrow$ light meson field

$\theta \longrightarrow$ the vacuum angle

The LECs: Σ, F, $L_{i=1-10}$, $\ldots$ parametrize the non-perturbative dynamics that is not determined by symmetry

The weak interactions responsible for weak decays as $\Delta S = 1$ can also be included

$$\mathcal{L}^{\Delta S=1} = \frac{G_F}{2\sqrt{2}} \sin\theta_C \cos\theta_C \sum_i C^i_W(\mu) O_i(\mu)$$

The $O_i's$ transform as $(27,1)$, $(8,1)$ or $(8,8)$ under $SU(3)_L \times SU(3)_R$: to leading order there are only four operators

$$\mathcal{L}^{\Delta S=1}_{\chi} = g_{(27,1)} t^{ij}_{kl} \left(U\partial_\mu U^\dagger\right)_{ik} \left(U\partial_\mu U^\dagger\right)_{jl} + g_{(8,8)} \tilde{t}^{ij}_{kl} U_{ik} U^r_{jl}$$
$$+ g^{(1)}_{(8,1)} \left(\partial_\mu U \partial_\mu U^\dagger\right)_{23} + g^{(2)}_{(8,1)} \left(MU + U^\dagger M^\dagger\right)_{23} + \ldots$$

The problem is that the matching should be done at $m_q \longrightarrow 0$ and not at $m_q \longrightarrow m_s$!

- Higher order effects are important around m_s
- Many couplings to determine: how many are necessary ?

Matching of QCD and χPT is important:

- Check the range of validity of χPT
- Get rid of the systematic error in chiral extrapolations
- Get the low energy couplings which determine light hadron physics

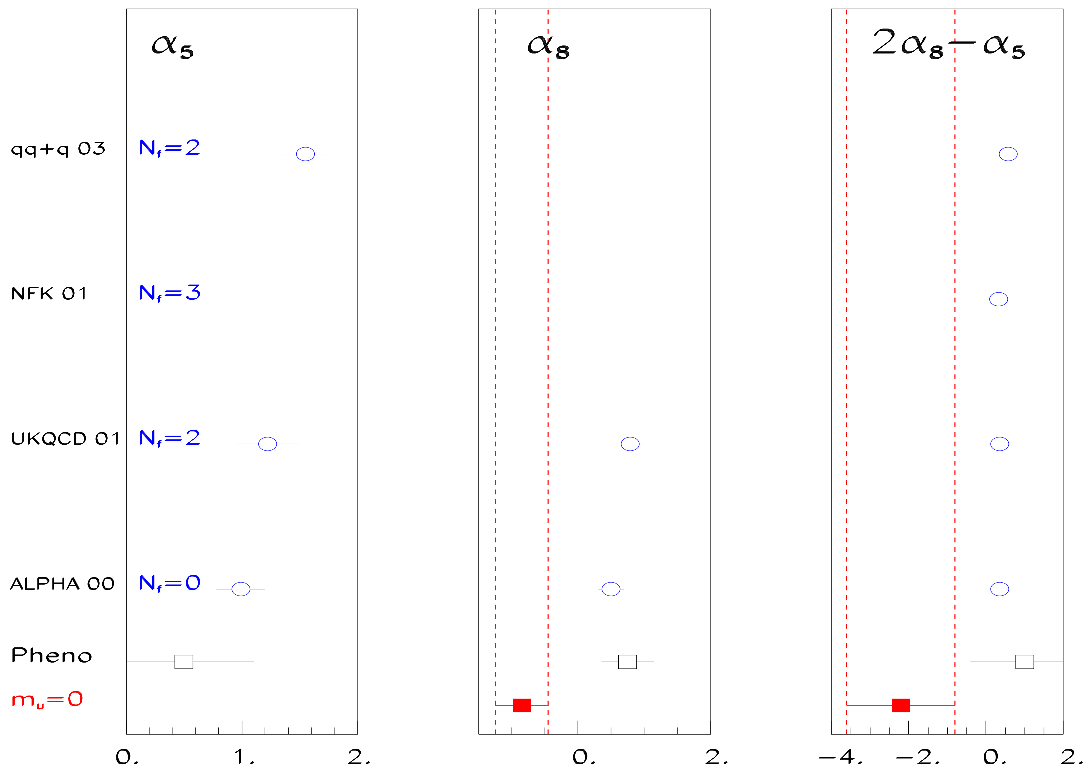
They are rather poorly known from phenomenology

i	$\alpha_i^r(M_\rho) \equiv 8(4\pi)^2 L_i^r(M_\rho)$	$O(N_C)$	Source
$2\alpha_1 - \alpha_2$	-0.8 ± 0.8	$O(1)$	K_{e4}, $\pi\pi \to \pi\pi$
α_2	1.7 ± 0.4	$O(N_C)$	K_{e4}, $\pi\pi \to \pi\pi$
α_3	-4.4 ± 1.4	$O(N_C)$	K_{e4}, $\pi\pi \to \pi\pi$
α_4	-0.4 ± 0.6	$O(1)$	Zweig rule
α_5	1.8 ± 0.6	$O(N_C)$	$F_K : F_\pi$
α_6	-0.25 ± 0.4	$O(1)$	Zweig rule
α_7	-0.5 ± 0.25	$O(1)$	GMO, α_5, α_8
α_8	1.1 ± 0.4	$O(N_C)$	M_ϕ, α_5
α_9	8.7 ± 0.9	$O(N_C)$	$\langle r^2 \rangle_V^\pi$
α_{10}	-6.9 ± 0.9	$O(N_C)$	$\pi \to e\nu\gamma$

$m_u = 0$ is connected to the value of $2\alpha_8 - \alpha_5$, which cannot be determined unambiguously in χPT

Kaplan, Manohar PRL56(1986); Leutwyler, NPB337 (1990)

$m_u = 0$?



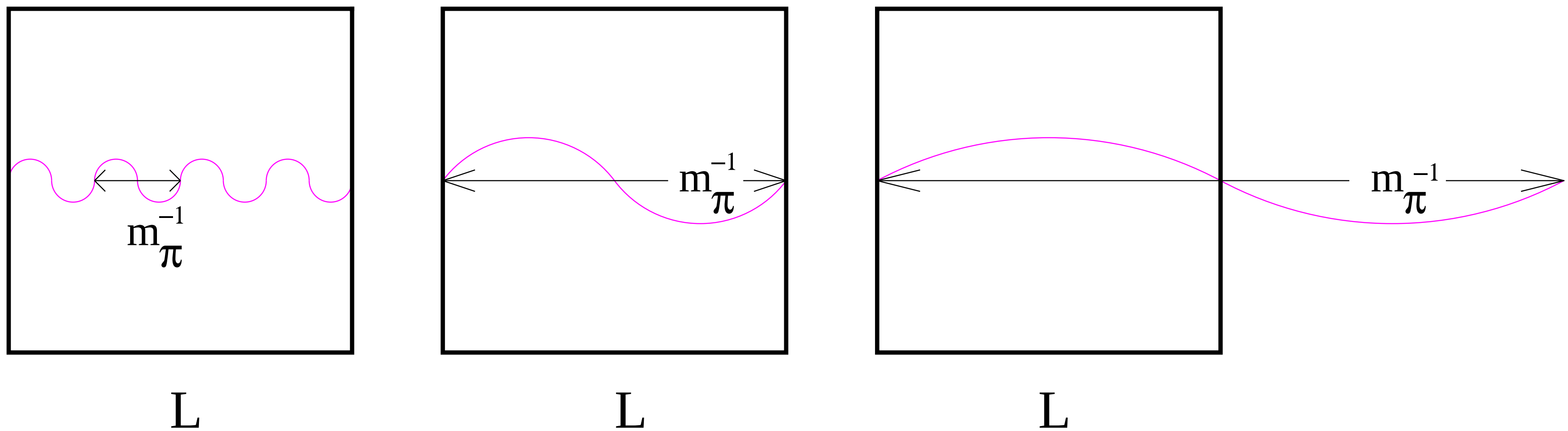
Heitger *et al* NPB588(2000); UKQCD, PLB518(2001);Nelson *et al*, PRL90 (2003); Farchioni *et al* hep-lat/0302011

- Good statistical accuracy, but still a large systematic effect from χ extrapolations
- A unreasonable large unknown systematic effect would be needed to accomodate $m_u = 0$!

New developments

Match QCD and χPT at $m_q \to 0$ in a finite volume $L\Lambda_{QCD} \gg 1$. In this case, finite volume effects are calculable within χPT provided $\Lambda_{QCD} L \gg 1$!

Light quarks in a box: $\Lambda_{QCD} L \gg 1$



$m_\pi L \gg 1$:
FS $\sim exp(-m_\pi L)$

$m_\pi L \sim 1$:
FS $\sim 1/(FL)^2$

$m_\pi L \ll 1$: if $F^2 m_\pi^2 V < 1$
ϵ-expansion

ϵ-expansion

If $M_P^2 F^2 V \simeq m\Sigma V \sim 1$ the zero mode becomes non-perturbative signalling that spontaneous symmetry breaking does not occur in a finite volume!

But the perturbative series can be reordered by factoring out the constant field configurations and treating them as collective variables:

$$U = U_0 U_\xi = U_0\, e^{\,i\,2\xi(x)/F} \qquad \int dx \xi(x) = 0$$

$$\mathcal{Z} = \int_{SU(N_f)} dU_o \int d\xi \;\; e^{-\mathcal{S}_\chi(U_o,\xi)}$$

Gasser, Leutwyler PLB188 (1987)

A convenient expansion for this regime is

$$\boxed{\frac{M_P}{4\pi F} \;\sim\; \left(\frac{p}{4\pi F}\right)^2 \;\sim\; \frac{1}{4\pi(LF)^2} \;\sim\; O(\epsilon^2)}$$

Implies a reordering of the chiral expansion:

$$\tilde{\mathcal{L}}_\chi = \tilde{\mathcal{L}}_\chi^{(0)} + \tilde{\mathcal{L}}_\chi^{(2)} + \ldots$$

$$\tilde{\mathcal{L}}_\chi^{(0)} = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U_\xi^\dagger \partial_\mu U_\xi\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[e^{i\theta/N_f} M U_0 + U_0^\dagger M^\dagger e^{-i\theta/N_f}\right]$$

NLO: $\chi \equiv MU \quad u_\mu \equiv i\partial_\mu U U^\dagger$

p-regime	ϵ-regime
$L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2$	$\surd$
$L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle$	$\surd$
$L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle$	$\surd$
$L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger\chi + \chi^\dagger U\rangle$	$\times$
$L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger\chi + \chi^\dagger U\right)\rangle$	$\times$
$L_6 \langle U^\dagger\chi + \chi^\dagger U\rangle^2$	$\times$
$L_7 \langle U^\dagger\chi - \chi^\dagger U\rangle^2$	$\times$
$L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle$	$\times$
$iL_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle$	$\surd$
$L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle$	$\surd$

A number of quantities have been computed to NLO in the ϵ-expansion: quark condensate, meson propagators in a θ vacuum and in fixed topology

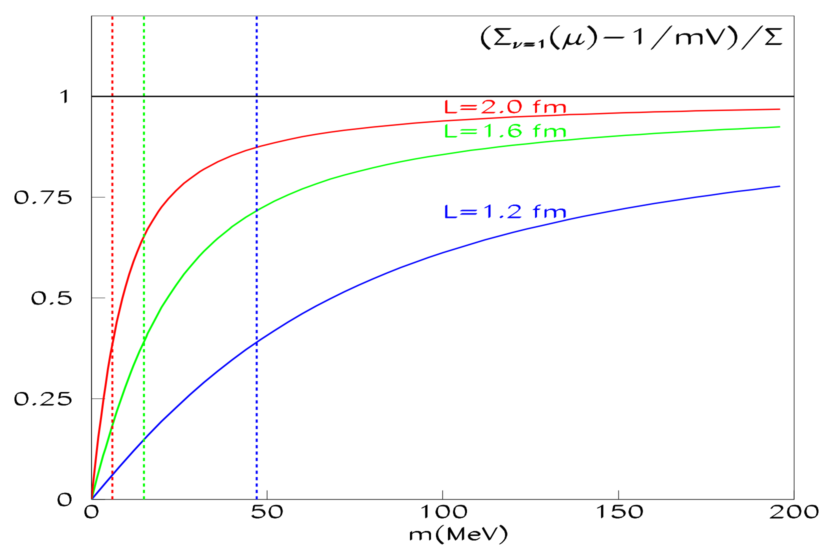
Hansen NPB345 (1990); Hansen, Leutwyler NPB350 (1991); P.H. Damgaard, *et al* NPB629(2002), 656(2003)

More recently three-point functions including the $\Delta S = 1$ Hamiltonian H., Laine JHEP 0301(2003)

The matching to lattice data should provide determinations of the low-energy couplings (QCD and weak) where chiral extrapolations are under control

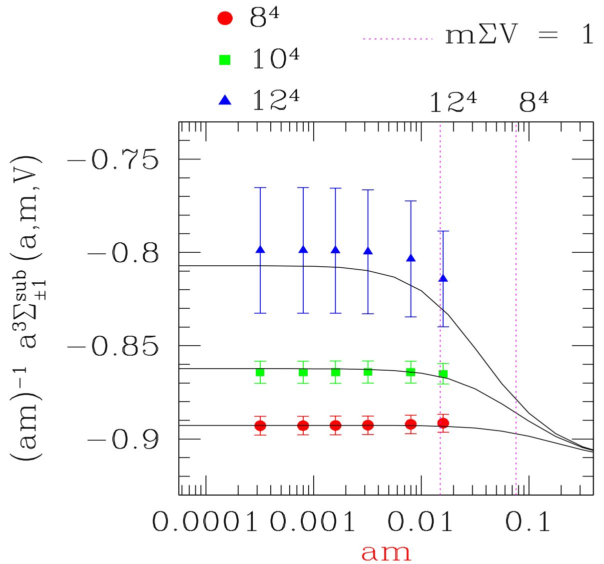
Example: $-\langle \bar{q}q \rangle_{\nu,m,V} = \frac{|\nu|}{mV} + \frac{m\Sigma^2 V}{2|\nu|} + O(m^2)$

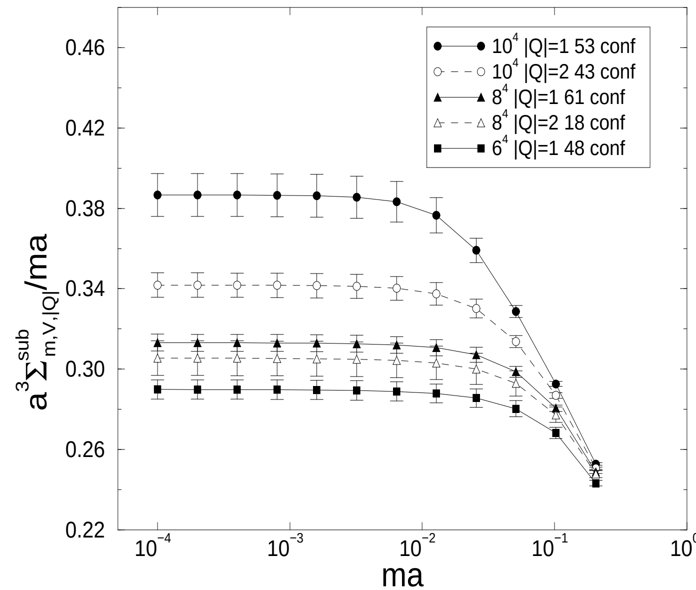
Gasser, Leutwyler, PLB188(1987); Damgaard, *et al*, NPB547 (1999)



Exploratory study $L \sim 1 - 1.5 fm$ with GW fermions in the quenched approximation

$$a^2 \Sigma_\nu^{sub}(a)/m = Z_S^{-1}(\alpha_s, a\mu_R) \left(\frac{a^2 \Sigma^2 V}{2|\nu|} - C_1 \right)$$





H., Jansen, Lellouch PLB469 (1999)

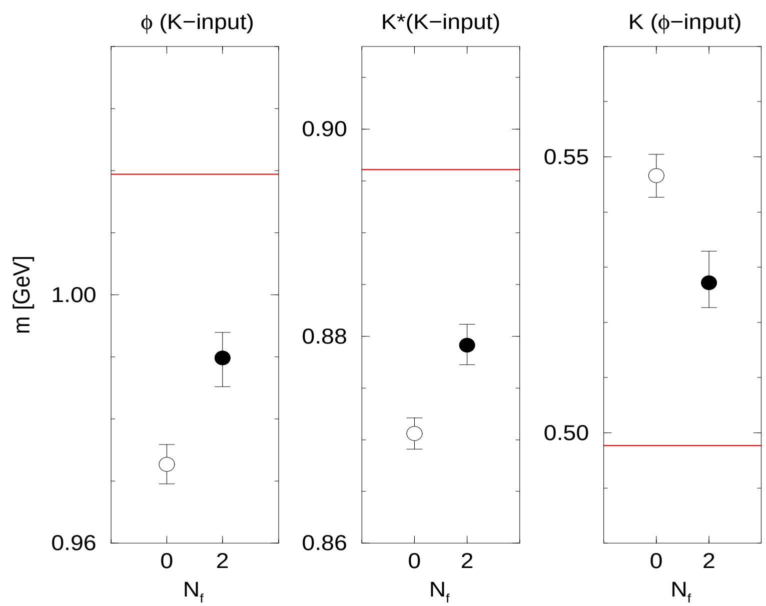
Hasenfratz *et al* NPB643 (2002)

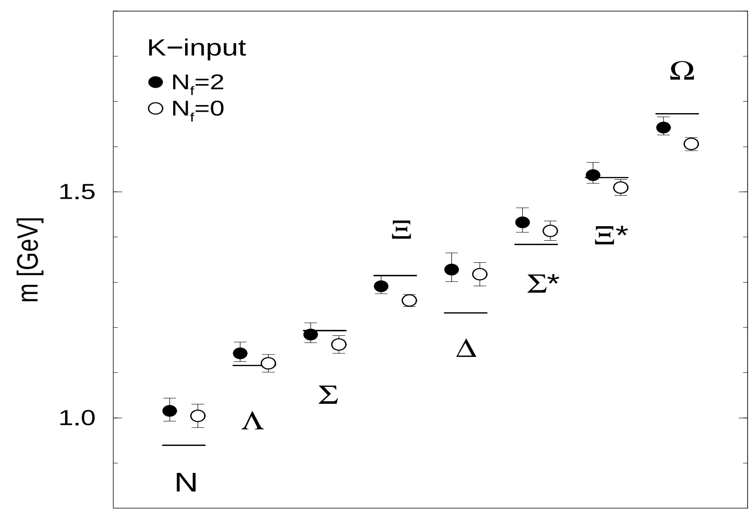
$$\Sigma_{FSS}^{\bar{M}S}(2\text{GeV}) = (268(12)\text{MeV})^3 \quad a^{-1} = 1.6\text{GeV}$$

A conceptually cleaner extraction of Σ.

The beast: unquenching

Several large collaborations with $O(1Tflop)$ computer power (CPPACS, JLQCD, MILC, UKQCD, ...) have produced in recent years many results beyond the quenched approximation with $N_f = 2$ mostly:





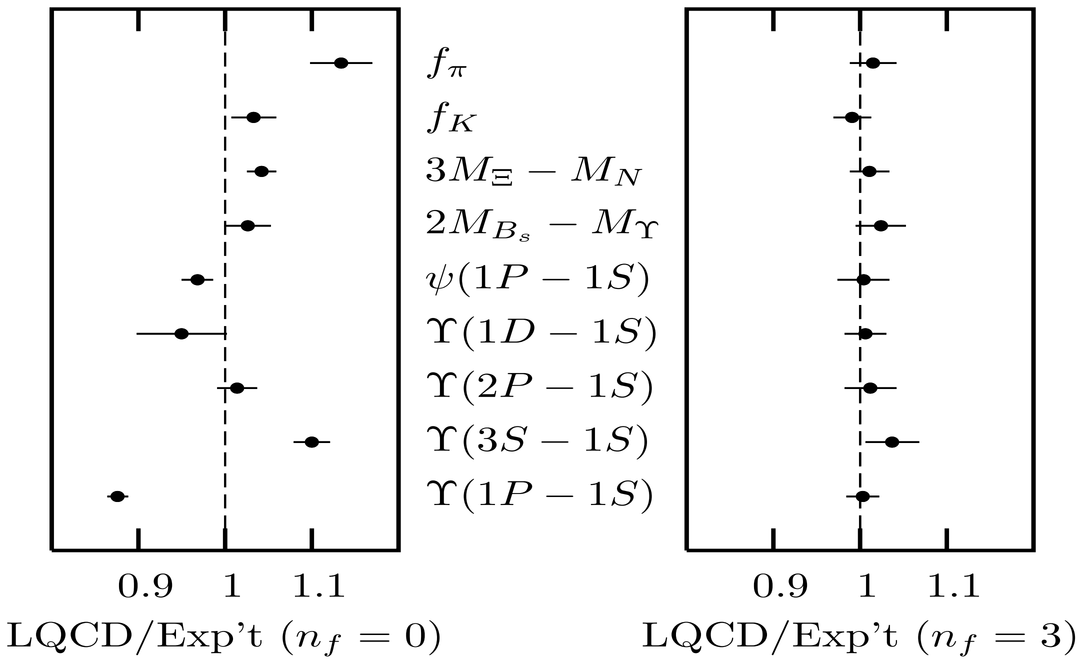
JLQCD, hep-lat/0212039

	$\mathcal{O}(N_f = 2)/\mathcal{O}(N_f = 0)$	coll.
m_s	0.7-0.8	CPPACS
F_{D_s,B_s}	1.1	CPPACS
		MILC
B_{B,B_s}	~ 1	JLQCD

Dominated by systematics:

$a \geq 0.09$fm, $\frac{m_\pi}{m_\rho} = 0.6 - 0.8$

Recent simulations with $N_f = 3$ and an "improved" staggered action at $a = 1/8 fm$ *corageously* confront experiment:



Davies *et al* hep-lat/0304004

These results are exciting ($m_\pi/m_\rho = 0.3!$) but questions of principle remain concerning the action

In the quenched approximation, systematics are under control for many quantities:

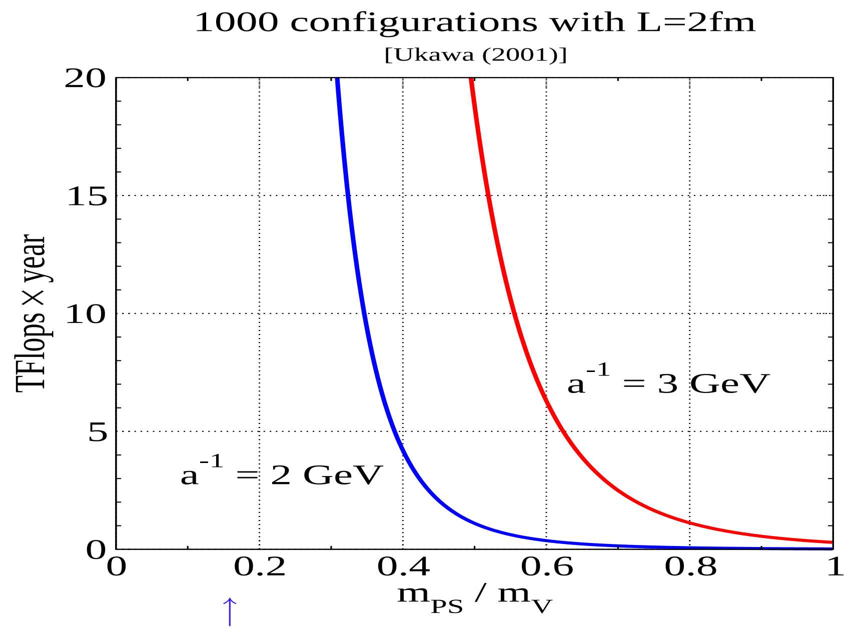
- Light hadron spectrum
- Strange quark physics: m_s, F_K
- Charm quark physics: m_c, F_{D_s}, quarkonium levels
- Running coupling
- ...

These are the benchmark for unquenched simulations in the future

The Berlin Lattice2001 wall...

Next $\longrightarrow$

Now $\longrightarrow$



$\uparrow$
phys

$$\frac{\#\text{operations}}{\text{fieldconfiguration}} \sim 3 \underbrace{\left[\frac{140MeV}{m_\pi}\right]^6}_{\downarrow} \left[\frac{L}{3fm}\right]^5 \left[\frac{0.1fm}{a}\right]^7 \text{Tflopsyear}$$

$$\text{Cost}(D^{-1}) \sim \frac{1}{m_q} \sim \frac{1}{m_\pi^2} \Rightarrow \quad \text{Cost}(HMC) \sim \tfrac{1}{m_q^2} \sim \tfrac{1}{m_\pi^4}(?)$$

Next generation of computers $O(10)$Tflops in a few years: $\frac{m_\pi}{m_\rho} \sim 0.4 - 0.6$, $a^{-1} = 2 - 3$ GeV (probably still far from physical point)

QCDOC, ApeNEXT, ...

If we do not get smarter probably another order of magnitude in computer power will be needed to arrive to the state-of-the-art quenched simulations of today

But...

- Large potential gain by improvements in the algorithms!
  Hasenbusch, Jasen hep-lat/0211024; Lüscher hep-lat/0304007
- Proliferation of *improved* actions: similar scaling, but might arrive there first. In many cases, questions of principle remain to be understood...

Conclusions

Lattice Field Theory is a mature and active field which has the best chance to answer non-perturbative questions in QCD and other gauge theories from first principles

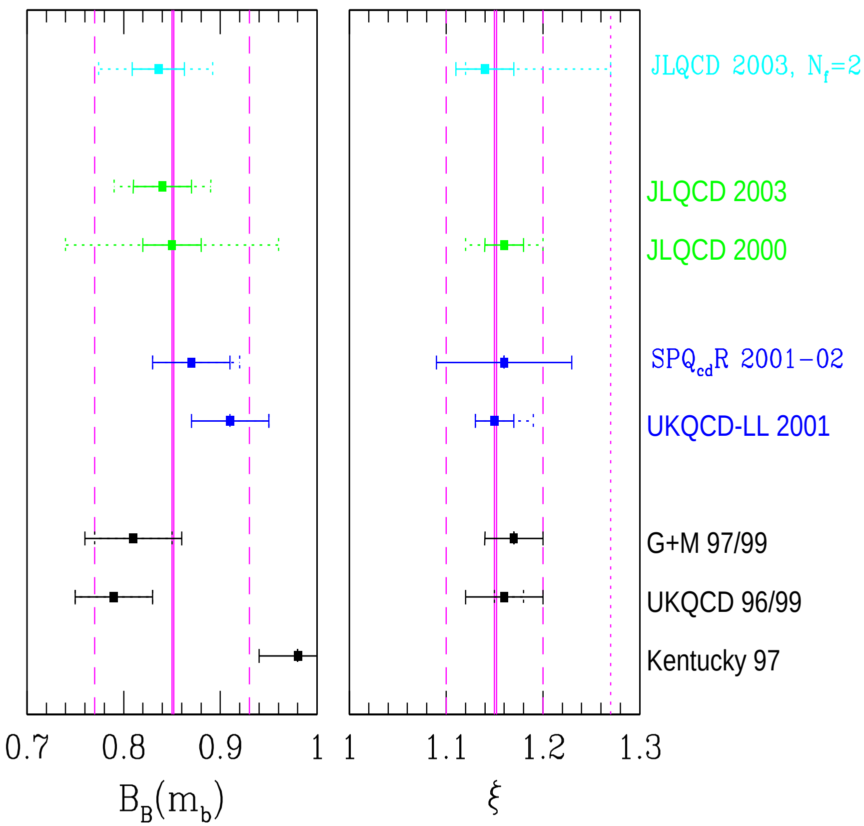
Progress in recent years has been remarkable thanks to

- Exponential improvement in computer power
- Improvements in algorithms, actions and the questions being asked

Many quantities have been computed in the *quenched* approximation to a few per cent accuracy with full control over systematic errors: light hadron spectrum, Λ, m_s, F_K, m_c, F_{D_s}, etc

A similar standard in unquenched simulations will take longer due to the Berlin wall...

New unquenched determination by JLQCD



Wittig's parallel talk